Brownian Motion, the Fredholm Determinant, and Time Series Analysis

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1. Quadratic functionals of Brownian motion (Bm)

Let $\{W(t)\}$ be standard Brownian motion (Bm) defined on $[0,1]$ and consider a quadratic functional of Bm given by

$$S = \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t), \tag{1}$$

where $K(s,t)$ is a symmetric, continuous, and nearly definite kernel defined on $[0,1]\times[0,1]$ with the definition of 'nearly definite kernel' given later.

The statistic S comes from

Theorem 1: Nabeya and Tanaka (1988). As $N \to \infty$, it holds that

$$S_N = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} B_N(j,k)\, \varepsilon_j\, \varepsilon_k = \frac{1}{N}\, \boldsymbol{\varepsilon}' B_N \boldsymbol{\varepsilon} \;\Rightarrow\; S, \tag{2}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, , \ldots, \varepsilon_N)'$ and $\{\varepsilon_j\} \sim$ i.i.d.$(0,1)$, whereas B_N is an $N \times N$ symmetric matrix whose (j,k)th element $B_N(j,k)$ satisfies

$$\lim_{N\to\infty} \max_{1\le j,k\le N} \left| B_N(j,k) - K\left(\frac{j}{N}, \frac{k}{N}\right)\right| = 0. \tag{3}$$

It can be seen from (3) that elements of B_N must be close to each other, which excludes sparse matrices like diagonal matrices. It is also clear that

$$S = \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) \overset{\mathcal{D}}{=} \int_0^1 \int_0^1 K(1-s, 1-t)\, dW(s)\, dW(t),$$

where $\overset{\mathcal{D}}{=}$ stands for distributional equivalence.

2. Fredholm determinant (FD)

Given a symmetric and continuous function $K(s,t)$, consider the following equation for λ and $f(t)$:

$$f(t) = \lambda \int_0^1 K(s,t) f(s)\, ds. \tag{4}$$

A value λ for which this integral equation possesses a nonvanishing continuous solution $f(t)$ is called an eigenvalue of $K(s,t)$; the corresponding solution $f(t)$ is called an eigenfunction for the eigenvalue λ. It is known that there exists at least one eigenvalue insofar as $K(s,t)$ is not identically equal to zero. Note also that every eigenvalue is real, whereas $\lambda = 0$ is certainly not an eigenvalue. The maximum number of linearly independent eigenfunctions corresponding to the eigenvalue λ is called the multiplicity of λ.

To define the FD of $K(s,t)$ we approximate the integral equation (4) by the algebraic system

$$f\left(\frac{k}{N}\right) = \frac{\lambda}{N}\sum_{j=1}^{N} K\left(\frac{j}{N},\frac{k}{N}\right) f\left(\frac{j}{N}\right) \qquad (k=1,\,\ldots,\,N),$$

or, in matrix notation,

$$\boldsymbol{f}_N = \frac{\lambda}{N}\,K_N\boldsymbol{f}_N \quad \Leftrightarrow \quad \left(I_N - \frac{\lambda}{N}\,K_N\right)\boldsymbol{f}_N = \mathbf{0}, \tag{5}$$

where $\boldsymbol{f}_N = [(f(k/N))]$ is an $N\times 1$ vector and $K_N = [(K(j/N,k/N))]$ is an $N\times N$ symmetric matrix. To obtain λ that satisfies (5), we consider

$$D_N(\lambda) = \left|I_N - \frac{\lambda}{N}\,K_N\right|. \tag{6}$$

$D_N(\lambda)$ is a polynomial of degree N in λ and its zeros give the reciprocals of the eigenvalues of K_N/N in matrix theory. The definition of $D_N(\lambda)$ in (6) gives a normalization $D_N(0) = 1$, which is independent of the matrix K_N. Then the FD of $K(s,t)$ is defined [Hochstadt (1973)] by

$$\begin{aligned} D(\lambda) &= \lim_{N\to\infty} D_N(\lambda) = \lim_{N\to\infty}\left|I_N - \frac{\lambda}{N}\,K_N\right| && (7)\\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^n d_n}{n!}\,\lambda^n, && (8)\end{aligned}$$

where

$$\begin{aligned} d_n &= \int_0^1\cdots\int_0^1 K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n\\ t_1 & \cdot & \cdot & \cdot & t_n\end{pmatrix} dt_1\cdots dt_n,\\ K\begin{pmatrix} t_1 & \cdot & \cdot & \cdot & t_n\\ t_1 & \cdot & \cdot & \cdot & t_n\end{pmatrix} &= \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdot\ \cdot\ \cdot & K(t_1,t_n)\\ \vdots & \vdots & & \vdots\\ K(t_n,t_1) & K(t_n,t_2) & \cdot\ \cdot\ \cdot & K(t_n,t_n)\end{vmatrix}.\end{aligned}$$

Note here that

$$\begin{aligned} d_1 &= \int_0^1 K(s,s)\,ds, && (9)\\ d_2 &= \int_0^1\int_0^1 \begin{vmatrix} K(s,s) & K(s,t)\\ K(s,t) & K(t,t)\end{vmatrix}\,ds\,dt\\ &= d_1^2 - \int_0^1\int_0^1 K^2(s,t)\,ds\,dt. && (10)\end{aligned}$$

The series in (8) converges for all λ [Hochstadt (1973)], that is, $D(\lambda)$ is an entire function of λ with $D(0)=1$. Then $D(\lambda)$ has an infinite product expansion [Hochstadt (1973, p.249)] expressed as

$$D(\lambda) = \exp\left\{-\lambda\int_0^1 K(t,t)\,dt\right\}\prod_{n=1}^{\infty}\left\{\left(1-\frac{\lambda}{\lambda_n}\right)\exp\left(\frac{\lambda}{\lambda_n}\right)\right\}, \tag{11}$$

where $\lambda_1, \lambda_2, \ldots$ are eigenvalues of $K(s,t)$ repeated as many times as their multiplicities. A much simpler product expansion for $D(\lambda)$ will be obtained shortly by imposing a condition on $K(s,t)$ in addition to symmetry and continuity.

It is recognized in (11) that the zeros of $D(\lambda)$ are eigenvalues of $K(s,t)$. In fact, we have [Hochstadt (1973, p.243)]

Theorem 2. Suppose that $K(s,t)$ is symmetric and continuous on $[0,1] \times [0,1]$. Then every zero of $D(\lambda)$ is an eigenvalue of $K(s,t)$, and in turn every eigenvalue of $K(s,t)$ is a zero of $D(\lambda)$.

It follows from Theorem 2 that $D(\lambda)$ has necessary and sufficient information about eigenvalues of the kernel $K(s,t)$. If $K(s,t)$ has an infinite number of eigenvalues, $K(s,t)$ is said to be nondegenerate; otherwise it is degenerate. If all of the eigenvalues are positive (negative), then $K(s,t)$ is said to be positive (negative) definite. This is equivalent to

$$\int_0^1 \int_0^1 K(s,t)\, g(s)\, g(t)\, ds\, dt \geq 0 \quad (\leq 0) \tag{12}$$

for any continuous function $g(t)$ defined on $[0,1]$. The kernel $K(s,t)$ is said to be nearly definite if all but a finite number of eigenvalues have the same sign. Any degenerate kernel is always nearly definite; so is the sum of a definite kernel and degenerate kernels.

The following theorem called Mercer's theorem is quite useful, not only for obtaining a simpler expression for $D(\lambda)$ in (11), but also for deriving distributions dealt with in subsequent discussions. For the proof, see Courant and Hilbert (1953, p.138) and Hochstadt (1973, p.91).

Theorem 3: Mercer's theorem. Let $K(s,t)$ be symmetric, continuous, and nearly definite in $[0,1] \times [0,1]$. Then $K(s,t)$ has the series expansion given by

$$K(s,t) = \sum_{n=1}^{\infty} \frac{f_n(s) f_n(t)}{\lambda_n}, \tag{13}$$

where $\{\lambda_n\}$ is a sequence of eigenvalues of $K(s,t)$ repeated as many times as their multiplicities, whereas $\{f_n(t)\}$ is an orthonormal sequence of eigenfunctions corresponding to λ_n and the series on the right side converges absolutely and uniformly to $K(s,t)$.

It is noticed from (13) that $K(s,t)$ and $K(1-s, 1-t)$ have the same set of eigenvalues, which implies that the two kernels have the same FD because of Theorem 4 described below. This can also be verified from (8) by checking the definition of d_n. It follows from Mercer's theorem that, if $K(s,t)$ is symmetric, continuous, and nearly definite,

$$\int_0^1 K(t,t)\, dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}, \tag{14}$$

where each eigenvalue is repeated as many times as its multiplicity and the sum converges absolutely. Thus we have, from (11),

$$\begin{aligned} D(\lambda) &= \exp\left\{ -\lambda \int_0^1 K(t,t)\, dt + \sum_{n=1}^{\infty} \frac{\lambda}{\lambda_n} \right\} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right). \end{aligned} \tag{15}$$

3. The characteristic function (c.f.) of the statistic S

We can now prove the Anderson-Darling theorem given by

Theorem 4: Anderson and Darling (1952). Consider the statistic

$$S = \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) \stackrel{\mathcal{D}}{=} \int_0^1 \int_0^1 K(1-s, 1-t)\, dW(s)\, dW(t), \tag{16}$$

where $K(s,t)$ is symmetric, continuous, and nearly definite. Then the c.f. of S is given by

$$\begin{aligned} \mathrm{E}\left(e^{i\theta S}\right) &= \mathrm{E}\left[\exp\left\{i\theta \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t)\right\}\right] \\ &= \left(D(2i\theta)\right)^{-1/2} && (17) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta}{\lambda_n}\right)^{-1/2}, && (18) \end{aligned}$$

where $D(\lambda)$ is the FD of $K(s,t)$, whereas $\{\lambda_n\}$ is a sequence of eigenvalues of $K(s,t)$ repeated as many times as their multiplicities.

Proof. We have, from Mercer's theorem,

$$\begin{aligned} S &= \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) = \int_0^1 \int_0^1 \sum_{n=1}^{\infty} \frac{f_n(s) f_n(t)}{\lambda_n}\, dW(s)\, dW(t) \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_0^1 f_n(t)\, dW(t)\right)^2 \stackrel{\mathcal{D}}{=} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n^2, \quad \{Z_n\} \sim \mathrm{NID}(0,1). \end{aligned}$$

Then we can compute the c.f. of S by using this last expression and (15) to obtain (17) and (18), which establishes the theorem.

Once the FD associated with the statistic S in (16) is derived, the computation of moments of S becomes easier, although there are various ways.

Theorem 5. For the statistic S in (16), its kth moment is given by

$$\begin{aligned} \mathrm{E}\left(S^k\right) &= \mathrm{E}\left[\left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n^2\right)^k\right] \\ &= \mathrm{E}\left[\left(\int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t)\right)^k\right] \\ &= \left.\frac{d^k \left(D(2i\theta)\right)^{-1/2}}{i^k\, d\theta^k}\right|_{\theta=0}. \end{aligned}$$

In particular, we have

$$
\begin{aligned}
\mathrm{E}(S) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \int_0^1 K(t,t)\,dt = d_1, && (19)\\
\mathrm{E}(S^2) &= 3d_1^2 - 2d_2, && (20)\\
\mathrm{Var}(S) &= 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = 2\int_0^1\int_0^1 K^2(s,t)\,ds\,dt && (21)\\
&= 2\left(d_1^2 - d_2\right), && (22)\\
\mathrm{E}(S^3) &= 15d_1^3 - 18d_1d_2 + 4d_3, \\
\mathrm{E}(S^4) &= 105d_1^4 - 180d_1^2d_2 + 48d_1d_3 + 36d_2^2 - 8d_4,
\end{aligned}
$$

where d_n $(n = 1, 2, 3, 4)$ are coefficients in the series expansion for $D(\lambda)$ given in (8) with d_1 and d_2 defined in (9) and (10), respectively.

Proof. Since it follows from Theorem 4 that

$$
S = \int_0^1\int_0^1 K(s,t)\,dW(s)\,dW(t) \overset{\mathcal{D}}{=} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} Z_n^2,
$$

the first and second equalities hold. The third equality comes from the property of the c.f. $\big(D(2i\theta)\big)^{-1/2}$. The first and second equalities in (19) are obvious and the third equality comes from

$$
\begin{aligned}
\mathrm{E}(S) &= \left.\frac{d\,(D(2\theta))^{-1/2}}{d\theta}\right|_{\theta=0} = \left.\frac{d}{d\theta}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n(2\theta)^n}{n!}\,d_n\right)^{-1/2}\right|_{\theta=0} \\
&= \left.\frac{d}{d\theta}\left(1 - 2\theta d_1 + \frac{4\theta^2}{2}d_2 - \frac{8\theta^3}{6}d_3 + \cdots\right)^{-1/2}\right|_{\theta=0} = d_1.
\end{aligned}
$$

We obtain (20) by computing $d^2\left(D(2\theta)\right)^{-1/2}/d\theta^2|_{\theta=0}$. The relation in (22) is obtained from (19) and (20). The first relation in (21) is obvious, whereas the second relation comes from (10) and (22). Similarly we can obtain the expressions for $\mathrm{E}(S^3)$ and $\mathrm{E}(S^4)$ by computing $d^k\left(D(2\theta)\right)^{-1/2}/d\theta^k|_{\theta=0}$ for $k = 3, 4$, which establishes the theorem.

4. A general procedure for deriving the FD

We have defined the FD $D(\lambda)$ in various ways [see (7), (8), and (15)]. It, however, is recognized that the computation of $D(\lambda)$ via these formulas is burdensome or impossible in general.

We present here a general method for obtaining the FD, which is to deal with a differential equation with some conditions equivalent to the original integral equation. We deal with two cases. One is the case where the integral equation is equivalent to the pth order homogeneous differential equation with p boundary conditions. The other is the case where the differential equation is a second order nonhomogeneous equation with two boundary conditions and some other extra restrictions.

Case 1: Suppose first that the differential equation is the pth order homogeneous equation that has a general solution given by

$$
f(t) = c_1\phi_1(t) + \cdots + c_p\phi_p(t), \tag{23}
$$

where $c_1, \ldots, c_p$ are arbitrary constants and $\phi_1(t), \ldots, \phi_p(t)$ are linearly independent continuous functions, whereas the boundary conditions are given by

$$M(\lambda)\boldsymbol{c} = \mathbf{0}, \quad M(\lambda) : p \times p, \quad \boldsymbol{c} = (c_1, c_2, \ldots, c_p)'. \tag{24}$$

Then we have the following theorem leading to the derivation of the FD.

Theorem 6: Kac, Kiefer, and Wolfowitz (1955), Nabeya and Tanaka (1988, 1990a). Suppose that the integral equation (4) is equivalent to the general solution (23) of a differential equation with the p boundary conditions $M(\lambda)\,\boldsymbol{c} = \mathbf{0}$, where $M(\lambda)$ and $\boldsymbol{c}$ are defined in (24). Then $|M(\lambda)| = 0$ is a necessary and sufficient condition for λ ($\neq 0$) to be an eigenvalue of $K(s,t)$ and the multiplicity ℓ_n of the eigenvalue λ_n is given by

$$\ell_n = p - \mathrm{rank}\,(M(\lambda_n))\,. \tag{25}$$

Note that the multiplicity is simply the dimension of a null space of $M(\lambda)$. The following theorem gives a set of sufficient conditions for a function of λ to be the FD.

Theorem 7: Nabeya and Tanaka (1988, 1990a). Let $K(s,t)$ be symmetric, continuous, and nearly definite in $[0,1] \times [0,1]$ and $\{\lambda_n\}$ be a sequence of eigenvalues of K. Suppose that $\tilde{D}(\lambda)$ is an entire function of λ with $\tilde{D}(0) = 1$. Then $\tilde{D}(\lambda)$ becomes the FD of K if

(i) every zero of $\tilde{D}(\lambda)$ is an eigenvalue of K, and in turn every eigenvalue of K is a zero of $\tilde{D}(\lambda)$;

(ii) $\tilde{D}(\lambda)$ can be expanded as

$$\tilde{D}(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right)^{\ell_n}, \tag{26}$$

where ℓ_n is the multiplicity of λ_n.

Case 2: Here we deal with the case where the integral equation (4) is equivalent to a second order nonhomogeneous differential equation with two boundary conditions and some other extra restrictions. For this purpose we consider the integral equation with the kernel defined by

$$K(s,t) = K_0(s,t) + G(s,t) + \sum_{k=1}^{q} \xi_k(s)\eta_k(t), \tag{27}$$

where $K_0(s,t) = 1 - \max(s,t)$ or $\min(s,t) - st$, whereas $G(s,t)$, $\xi_k(s)$ and $\eta_k(t)$ ($k = 1, \ldots, q$) are deterministic functions that satisfy

i) $\partial^2 G(s,t)/\partial t^2 = 0$, whereas $\xi_k(s)$ and $\eta_k(t)$ ($k = 1, \ldots, q$) are continuous and each set is linearly independent in the space $C[0,1]$.

ii) $\eta_k''(t)$ ($k = 1, \ldots, q$) are linearly independent.

Then we are led to the second order nonhomogeneous differential equation of the form

$$f''(t) + \lambda f(t) = \lambda \sum_{k=1}^{q} a_k \eta_k''(t), \tag{28}$$

where

$$a_k = \int_0^1 \xi_k(s) f(s)\, ds \qquad (k = 1, \ldots, q), \tag{29}$$

and we have two appropriate boundary conditions.

The general solution to (28) is

$$f(t) = c_1 \cos \sqrt{\lambda}\, t + c_2 \sin \sqrt{\lambda}\, t + \sum_{k=1}^{q} a_k g_k(t), \tag{30}$$

where $g_k(t)$ is a special solution of

$$g_k''(t) + \lambda g_k(t) = \lambda \eta_k''(t). \tag{31}$$

The q equations in (29) and two boundary conditions yield a system of $q+2$ linear homogeneous equations $M(\lambda)\, \boldsymbol{c} = \mathbf{0}$ in $\boldsymbol{c} = (a_1, \ldots, a_q, c_1, c_2)'$. Then we compute $|M(\lambda)|$ to obtain a candidate for the FD, determining the multiplicity ℓ_n of each eigenvalue λ_n by $\ell_n = q + 2 - \mathrm{rank}(M(\lambda_n))$ from Theorem 6. Following Theorem 7 we can find a candidate for the FD to ensure that it really is the FD.

There are various kernels that can be handled by the above methodologies. These include goodness of fit test statistics [Watson (1961)], test statistics for parameter constancy [MacNeill (1974), Nabeya and Tanaka (1988)], AR unit root test statistics [Nabeya and Tanaka (1990a, 1990b), Tanaka (2017)], MA unit root test statistics [Tanaka (1990)], multiple unit root statistics [Tanaka (1999)], fractional unit root statistics [Tanaka (2017)] and so on.

5. Darling's Formula and its extensions

The general procedure discussed in the last section enables us to obtain the FD for a broad class of kernels, but there are some kernels to which the above methodologies are complicated to apply. Here we suggest simple and efficient methods for computing the FDs of kernels that satisfy some restrictions. We start with the following theorem.

Theorem 8: Darling (1955), Sukhatme (1973). Consider

$$K(s,t) = K_0(s,t) - \sum_{j=1}^{m} \psi_j(s)\psi_j(t), \tag{32}$$

where $K_0(s,t)$ is a symmetric, continuous, and positive definite kernel whose FD is $D_0(\lambda)$, and $\psi_j(t)$ $(j = 1, \ldots, m)$ are continuous and linearly independent functions. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of $K_0(s,t)$, where the multiplicity of each eigenvalue is unity, and let $f_1(t), f_2(t), \ldots$ be the corresponding orthonormal eigenfunctions. Then the FD of $K(s,t)$ is given by

$$D(\lambda) = D_0(\lambda)\, |P(\lambda)|, \tag{33}$$

where $P(\lambda)$ is the $m \times m$ symmetric matrix whose (j,k)th elements are

$$\begin{aligned} P_{jj}(\lambda) &= 1 + \lambda \sum_{n=1}^{\infty} \frac{b_{jn}^2}{1 - \lambda/\lambda_n}, \quad P_{jk}(\lambda) = \lambda \sum_{n=1}^{\infty} \frac{b_{jn} b_{kn}}{1 - \lambda/\lambda_n} \quad (j \neq k), \\ b_{jn} &= \int_0^1 \psi_j(t) f_n(t)\, dt. \end{aligned}$$

The expression for $D(\lambda)$ in (33) is simple, but its computation deals with the infinite sumes contained in $P_{jk}(\lambda)$. To compute the infinite sums easily, we become more specific about $K_0(s,t)$ and $\psi(t)$. Then we have

Theorem 9: Tanaka (2024). Consider the kernel defined by

$$K(s,t) = \min(s,t) - st - \sum_{j=1}^{m} \psi_j(s)\psi_j(t), \quad \psi_j(0) = \psi_j(1) = 0, \tag{34}$$

where $\psi_j(t)$ $(j = 1, \ldots, m)$ are continuously differentiable. Then its FD is given by

$$D(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\, |P(\lambda)|, \tag{35}$$

where the (j,k)th elements $P_{jk}(\lambda)$ of the $m \times m$ symmetric matrix $P(\lambda)$ are

$$\begin{aligned}
P_{jj}(\lambda) &= 1 - \frac{2\sqrt{\lambda}}{\sin\sqrt{\lambda}} \int_0^1 \left(\int_0^t \psi_j'(s) \cos\sqrt{\lambda}s\, ds \right) \psi_j'(t) \cos\sqrt{\lambda}(1-t)\, dt, \\
P_{jk}(\lambda) &= -\frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}} \int_0^1 \int_0^1 \psi_j'(s)\psi_k'(t) L_1(s,t)\, ds\, dt \qquad (j \neq k), \\
L_1(s,t) &= \begin{cases} \cos\sqrt{\lambda}s\, \cos\sqrt{\lambda}(1-t) & (s \leq t), \\ \cos\sqrt{\lambda}t\, \cos\sqrt{\lambda}(1-s) & (s \geq t). \end{cases}
\end{aligned}$$

Specifying the kernel $K_0(s,t)$ in (32) as $1 - \max(s,t)$, we can establish the following theorem.

Theorem 10: Tanaka (2024). Consider the kernel defined by

$$K(s,t) = 1 - \max(s,t) - \sum_{j=1}^{m} \psi_j(s)\psi_j(t), \quad \psi_j(1) = 0, \tag{36}$$

where $\psi_j(t)$ $(j = 1, \ldots, m)$ are continuously differentiable. Then its FD is given by

$$D(\lambda) = \cos\sqrt{\lambda}\, |P(\lambda)|, \tag{37}$$

where the (j,k)th elements $P_{jk}(\lambda)$ of the $m \times m$ symmetric matrix $P(\lambda)$ are

$$\begin{aligned}
P_{jj}(\lambda) &= 1 + \frac{2\sqrt{\lambda}}{\cos\sqrt{\lambda}} \int_0^1 \left(\int_0^t \psi_j'(s) \sin\sqrt{\lambda}s\, ds \right) \psi_j'(t) \cos\sqrt{\lambda}(1-t)\, dt, \\
P_{jk}(\lambda) &= \frac{\sqrt{\lambda}}{\cos\sqrt{\lambda}} \int_0^1 \int_0^1 \psi_j'(s)\psi_k'(t) L_2(s,t)\, ds\, dt \qquad (j \neq k), \\
L_2(s,t) &= \begin{cases} \sin\sqrt{\lambda}s\, \cos\sqrt{\lambda}(1-t) & (s \leq t), \\ \sin\sqrt{\lambda}t\, \cos\sqrt{\lambda}(1-s) & (s \geq t). \end{cases}
\end{aligned}$$

6. Concluding remarks

Here we have concentrated on purely quadratic functionals of Bm:

$$S = \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t),$$

and discussed how to compute the c.f. of S by deriving the FD of $K(s,t)$.

Similarly we can deal with a ratio statistic :

$$R = \frac{\displaystyle\int_0^1 \int_0^1 K_N(s,t)\, dW(s)\, dW(t)}{\displaystyle\int_0^1 \int_0^1 K_D(s,t)\, dW(s)\, dW(t)},$$

where $K_D(s,t)$ is positive definite and $K_N(s,t)$ is degenerate. Then we consider

$$\begin{aligned} P(R \le x) &= P\left(\int_0^1 \int_0^1 \big[xK_D(s,t) - K_N(s,t)\big]\, dW(s)\, dW(t) \ge 0\right) \\ &= P\left(\int_0^1 \int_0^1 K(s,t;x)\, dW(s)\, dW(t) \ge 0\right), \end{aligned}$$

and the distribution function of R can be computed from the FD of $K(s,t;x)$ and Imhof's formula.

It is also sometimes the case that we have to deal with quadratic functionals of Bm added with linear or bilinear functionals. For example, we have

$$\begin{aligned} S_1 &= \int_0^1 \big(X(t) + m(t)\big)^2\, dt, \\ S_2 &= \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) + \int_0^1 m(t)\, dW(t), \\ S_3 &= \int_0^1 \big(X(t) + Zm(t)\big)^2\, dt, \\ S_4 &= \int_0^1 \int_0^1 K(s,t)\, dW(s)\, dW(t) + Z\int_0^1 m(t)\, dW(t), \end{aligned}$$

where $X(t)$ is a zero-mean Gaussian process, whereas $m(t)$ is an ordinary continuous function and $Z \sim \mathrm{N}(0,1)$ that is independent of $W(t)$.

We also consider the distribution of

$$S_H = \int_0^1 B_H^2(t)\, dt \stackrel{\mathcal{D}}{=} \int_0^1 \int_0^1 \frac{1}{2}\big[s^{2H} + t^{2H} - |s-t|^{2H}\big]\, dW(s)\, dW(t),$$

where $\{B_H(t)\}$ is fractional Brownian motion (fBm) defined on $[0,1]$, and H is the Hurst index $(0 < H < 1)$. The FD of $K_H(s,t) = \dfrac{1}{2}\big[s^{2H} + t^{2H} - |s-t|^{2H}\big]$ remains to be derived, for which Tanaka (2014) gives an approximation.

Tanaka (2024) discusses in detail the above problems together with various statistical examples.

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