Subexponentiality of densities of infinitely divisible distributions on the whole real line

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Abstract

We show the equivalence of three properties for an infinitely divisible distribution: the subexponentiality of the density, the subexponentiality of the density of its Lévy measure and the tail equivalence between the density and its Lévy measure density, under monotonic-type assumptions on the Lévy measure density. The key assumption is that tail of the Lévy measure density is asymptotic to a non-increasing function or is almost decreasing. Our conditions are natural and cover a rather wide class of infinitely divisible distributions. Several significant properties for analyzing the subexponentiality of densities have been derived such as closure properties of [convolution, convolution roots and asymptotic equivalence] and the factorization property. Moreover, we illustrate that the results are applicable for developing the statistical inference of subexponential infinitely divisible distributions which are absolutely continuous.

Introduction

Let f, g be probability density functions on $\mathbb{R}$ and denote by $f * g$ the convolution of f and g:

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

and denote by f^{*n} the nth convolutions with itself. Throughout the paper, for functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}_+$, $\alpha(x) \sim \beta(x)$ means that $\lim_{x\to\infty} \alpha(x)/\beta(x) \to 1$. We study the following characteristics for densities.

Definition 0.1. (i) *f is (right-side) long-tailed, denoted by $f \in \mathcal{L}$, if there exists $x_0 > 0$ such that $f(x) > 0$, $x \ge x_0$ and for any fixed $y > 0$ $f(x+y) \sim f(x)$.*
(ii) *f is (right-side) subexponential on $\mathbb{R}$, denoted by $\mathcal{S}$, if $f \in \mathcal{L}$ and $f^{*2}(x) \sim 2f(x)$.*
(iii) *f with dist. F is weakly (right-side) subexponential on $\mathbb{R}$, denoted by $\mathcal{S}_+$, if $f \in \mathcal{L}$ and the function $f_+(x) = \mathbf{1}_{\mathbb{R}_+}(x) f(x)/\overline{F}(0)$, $x \in \mathbb{R}$ is subexponential, i.e. $f_+ \in \mathcal{S}$. Here $\overline{F}(x) = 1 - F(x)$.*

Definition 0.2. (i) *We say that a density $f : \mathbb{R} \to \mathbb{R}_+$ is asymptotic to a non-increasing function (a.n.i. for short) if f is locally bounded and positive on $[x_0, \infty)$ for some $x_0 > 0$, and*

$$\sup_{t\ge x} f(t) \sim f(x) \quad \text{and} \quad \inf_{x_0 \le t \le x} f(t) \sim f(x). \tag{0.1}$$

(ii) *We say that a density $f : \mathbb{R} \to \mathbb{R}_+$ is almost decreasing (al.d. for short) if there exists $x_0 > 0$ and $K > 0$ such that*

$$f(x+y) \le K f(x), \quad \text{for all } x > x_0,\ y > 0.$$

Notice that the al.d. property includes the a.n.i. property, and the latter is satisfied by the regularly varying functions with negative indices.

We will investigate properties of the above sort, particularly on infinitely divisible distributions μ on $\mathbb{R}$. The characteristic function (ch.f.) of μ is

$$\widehat{\mu}(z) = \exp\Big\{ \int_{-\infty}^{\infty} (e^{izy} - 1 - izy\mathbf{1}_{\{|y|\le 1\}})\nu(dy) + iaz - \frac{1}{2}b^2 z^2 \Big\}, \tag{0.2}$$

where $a \in \mathbb{R}$, $b \ge 0$ and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty$. Throughout this paper, we always assume that the Lévy measure ν of μ has a density, and we denote by $\mathrm{ID}(\mathbb{R})$ the class of all infinitely divisible distributions on $\mathbb{R}$.

Main contents

We separate the cases depending on weather $\nu(\mathbb{R}) < \infty$ or $\nu(\mathbb{R}) = \infty$. Note that we use notation g also for the (non-proper) density of a Lévy measure. The followings are the results for the absolutely continuous case ($\nu(\mathbb{R}) = \infty$).

Theorem 0.3. *Let $\mu \in \mathrm{ID}(\mathbb{R})$ with $\nu(dx) = g(x)dx$ such that $\nu(\mathbb{R}) = \infty$. Let $f_0(x)$ be a density of $\mu_0 \in \mathrm{ID}(\mathbb{R})$ with $a = b = 0$ and $\nu(dx) = \mathbf{1}_{\{|x|\le 1\}} g(x)dx$. Suppose that $g_1(x) = \mathbf{1}_{\{x>1\}} g(x)/\nu((1,\infty))$ is bounded, and there exists $\gamma > 0$ such that*

$$\lim_{x\to\infty} e^{\gamma x} f_0(x) = 0. \tag{0.3}$$

For a density f of μ we consider the following properties.

$$\begin{aligned} &\text{(i)} && f \in \mathcal{S}_+ \quad \textit{and} \quad f \textit{ is al.d.} \\ &\text{(ii)} && g_1 \in \mathcal{S}_+ \\ &\text{(iii)} && g_1 \in \mathcal{L} \quad \& \quad \lim_{x\to\infty} f(x)/g_1(x) = \nu((1,\infty)). \end{aligned}$$

(a) If g is a.n.i., then we can choose f such that (i), (ii) *and* (iii) *are equivalent.*
(b) If g is al.d., then we can choose f such that (ii) $\Leftrightarrow$ (iii) *implies* (i).

We could remove several conditions in Theorem 0.3 by assuming the absolute integrability of the spectrally positive part $\widehat{\mu}_+(z)$.

Theorem 0.4. *Let $\mu \in \mathrm{ID}(\mathbb{R})$ with and $\nu(dx) = g(x)dx$ such that $g_1(x)$ is bounded. Suppose that $\int_{-\infty}^{\infty} |\widehat{\mu}_+(z)|dz < \infty$, which implies $\int_{-\infty}^{\infty} |\widehat{\mu}(z)|dz < \infty$, so that μ has a bounded continuous density f. Then the following relations hold between the properties* (i), (ii) *and* (iii) *of Theorem 0.3.*
(a) If g is a.n.i., then we can choose f such that (i), (ii) *and* (iii) *are equivalent.*
(b) If g is al.d., then we can choose f such that (ii) $\Leftrightarrow$ (iii) *implies* (i).

We apply our results to the consistency proof of the maximum likelihood estimation (MLE for short) for $\mu \in \mathrm{ID}(\mathbb{R})$ which is absolutely continuous. For simplicity we put $a = b = 0$ in $\widehat{\mu}(z)$ of (0.2) and assume that the spectrally positive part $\widehat{\mu}_+(z)$ is absolutely integrable.

Let $f(x;\theta)$ be the density of μ with θ a parameter vector and $g(x;\theta)$ be a density of the corresponding Lévy measure ν. Let $(X_1, \dots, X_n)$ be a random sample from $f(x;\theta_0)$ with $\theta_0 \in \Theta$ where Θ is a compact parameter space. Define the likelihood function

$$M_n(\theta) = n^{-1} \sum_{i=1}^{n} \log f(X_i;\theta).$$

MLE $\hat{\theta}_n$ maximizes the function $\theta \mapsto M_n(\theta)$. We say that a function $\alpha(x;\theta)$ is identifiable if $\alpha(\cdot\,;\theta) \ne \alpha(\cdot\,;\theta')$ every $\theta \ne \theta' \in \Theta$, i.e. $\alpha(x;\theta) \overset{a.e.}{=} \alpha(x;\theta')$ does not hold. For convenience, we only consider the symmetric or positive-half case, but we can easily generalize the result in the non-symmetric two-sided case. We use the function g_1 defined in Theorem 0.4.

Proposition 0.5. *Let $\mu \in \mathrm{ID}(\mathbb{R})$ given by* (0.2) *with $a = b = 0$ such that $\widehat{\mu}_+(z)$ is absolutely integrable. Let $g(x;\theta)$ be a symmetric or positive-half density of ν. Suppose* (i) : *$g(x;\theta)$ is identifiable, $\theta \mapsto g(x;\theta)$ is continuous in θ for every x, and $\int (\sup_{\theta\in\Theta} |\log g_1(x;\theta)|) g_1(x;\theta_0)dx < \infty$ with Θ a compact set such that $\theta_0 \in \Theta$. Suppose* (ii) : *$g_1(x;\theta)$ is bounded and a.n.i., and $g_1 \in \mathcal{S}$. Then MLE $\hat{\theta}_n$ satisfies $\hat{\theta}_n \xrightarrow{p} \theta_0$.*

References

[1] Matsui, M. (2022) Subexponentialiy of densities of infinitely divisible distributions (submitted).