On Asymptotic Distribution in Martingale Convergence of Supercritical Branching Processes with Poissonian offsprings

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Abstract

A branching process is a mathematical model of Erdos-Renyi random graphs. For a normalized process, a martingale convergence theorem holds. However, its asymptotic distribution has not been known so far. We propose a characterization of the distribution via analysis of a functional equation for the Laplace transform of the distribution. It turns out that a numerical analysis mostly coincides well with the theoretical characterization. Applications of the Branching process include a biological population, nuclear chain reactions, and the spread of computer software viruses in common. Mathematical models of these applications play a central role in figuring out the main process and predicting future extensions.

1 Background and Problem

Let X be a random variable (r.v.) taking nonnegative integer values $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and let $\boldsymbol{p} = \{p_k \,|\, k \in \mathbb{N}_0\}$ be its distribution. Then, a branching process (BP) $\{Z_n | n \in \mathbb{N}_0\}$ is defined by

$$Z_n = \sum_{i=0}^{Z_{n-1}} X_{n,i}$$

where $\{X_{n,i} \,|\, i \in \mathbb{N}_0\}$ is an iid sequence of r. v.s with $X_{n,i} \sim X$ for each $n \in \mathbb{N}_0$. That is, each of the i-th individuals in $(n-1)$-th generation produces $X_{n,i}$ offspring according to the same distribution $\boldsymbol{p}$ independently of each other, to form the n-th generation individuals. We assume that $Z_0 = 1$. In this case, $Z_1 \sim X$. Let $\mathbb{P}$ be the distribution of $\{Z_n\}$. For the BP, the extinction probability and the survival probability are defined by

$$\eta = \mathbb{P}(\exists n : Z_n = 0) \qquad \text{and}$$
$$\zeta = \mathbb{P}(Z_n > 0, \forall n \geq 0) = 1 - \eta, \tag{1.1}$$

respectively. When $m = \mathbb{E}[Z_1] \leq 1$, the population dies out with extinction probability $\eta = 1$. If $m > 1$, it is called supercritical. In the latter case, the extinction probability should be $\eta < 1$. Let

$W_n = Z_n/m^n, n = 0, 1, 2, \cdots$. Then, $\{W_n\}$ is known [2] to form a martingale and converges to a random variable W_∞ on $\mathbb{R}_+ = [0, \infty)$ a.s. as $n \to \infty$. We should recall here a profound theorem of Kesten-Stigum [2] in supercritical case that states $\mathbb{P}(W_\infty = 0) = \eta$ and $\mathbb{E}[W_\infty] = 1$ if and only if $\mathbb{E}[X \log X] < \infty$. Thus, an exact correspondence holds for the asymptotic r. v. W_∞.

However, not much is known about the distribution of W_∞ so far [1]. The objective of the paper is to characterize the distribution, by analyzing a functional equation that holds for the Laplace transform of the distribution, given below.

2 Analysis of the Distribution

Let $f : [0, 1] \mapsto [0, 1]$ be the probability generating function of $\boldsymbol{p}$: $f(s) = \sum_{k \in \mathbb{N}_0} p_k \, s^k$, $s \in [0, 1]$. W_∞ is known to have a density function on $(0, \infty)$ [2, Corollary 12.1], which we denote by $w : (0, \infty) \mapsto \mathbb{R}_+$, while W_∞ has a point mass at the origin, $P(W_\infty = 0) = \eta$ [2, Theorem 6.2]. Thus, we may write $W_\infty \sim \eta\delta(x) + w(x)$ on $x \in \mathbb{R}_+$.

Let $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the Laplace transform of W_∞,

$$(2.2) \qquad \varphi_0(u) = \mathbb{E}[e^{-uW_\infty}] = \int_{\mathbb{R}_+} e^{-ux} w(x) \, dx = \eta + \varphi(u), \qquad \varphi(u) = \int_{(0,\infty)} e^{-ux} w(x) \, dx.$$

For φ and f, the following functional equation, called *Abel's equation*, is known to hold [2, Sections 1.6 and 1.10]:

$$(2.3) \qquad \varphi_0(u) = f\left(\varphi_0\left(\frac{u}{m}\right)\right) = \sum_{k \in \mathbb{N}_0} p_k \, \varphi_0^k\left(\frac{u}{m}\right), \qquad u \in \mathbb{R}_+.$$

Especially, taking $u = 0$ or ∞ in (2.3) corresponds in general to the two solutions of the equation $s = f(s)$ with $s = 1 = \varphi_0(0)$ or $s = \eta = \varphi_0(\infty)$, respectively.

In this paper, we assume that the offspring distribution $\boldsymbol{p}$ is Poissonian, $X \sim$Pois(λ) for a $\lambda > 1$: $p_k = \dfrac{\lambda^k}{k!} e^{-\lambda}$. Then, taking especially $m = \lambda$ in (2.3), the Abel's equation reads

$$(2.4) \qquad \varphi_0(u) = \exp\left[\lambda\left\{\varphi_0\left(\frac{u}{\lambda}\right) - 1\right\}\right], \qquad u \in \mathbb{R}_+,$$

or, what is the same,

$$(2.5) \qquad \varphi(u) + \eta = \exp\left[\lambda\left\{\varphi\left(\frac{u}{\lambda}\right) - \zeta\right\}\right], \qquad u \in \mathbb{R}_+.$$

Thus, we characterize the distribution $w(x)$ that satisfies the Poisson-Abel equation (2.5), in this paper.

Taking the differentiation of (2.5), we have

$$(2.6) \qquad \varphi'(u) = \varphi'\left(\frac{u}{\lambda}\right)\left(\varphi(u) + \eta\right).$$

In the McLaurin expansion[1] of $\varphi(u)$, $\varphi(u) = \sum_{n\in\mathbb{N}_0} \frac{\varphi^{(n)}(0)}{n!}\, u^n$, we calculate $\varphi^{(n)}(0)$, $n \in \mathbb{N}$ through (2.6):

$$
\begin{aligned}
\varphi^{(n)}(u) &= \big(\,\varphi'(u)\,\big)^{(n-1)} = \Big\{\, \varphi'\Big(\frac{u}{\lambda}\Big)\, \Big(\varphi(u)+\eta\Big)\,\Big\}^{(n-1)} \\
&= \lambda^{-(n-1)}\varphi^{(n)}\Big(\frac{u}{\lambda}\Big)\, \Big(\varphi(u)+\eta\Big) + \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)}\varphi^{(n-l)}\Big(\frac{u}{\lambda}\Big)\, \varphi^{(l)}(u)
\end{aligned}
\tag{2.7}
$$

by Leibniz's formula, so that, taking $u = 0$ especially,

$$
\Big(1-\lambda^{-(n-1)}\Big)\varphi^{(n)}(0) = \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)}\varphi^{(n-l)}(0)\, \varphi^{(l)}(0), \tag{2.8}
$$

since $1 = \varphi_0(0) = \eta + \varphi(0)$ by (2.2). Here, let us introduce

$$
\rho \triangleq -\frac{\lambda}{\lambda-1}\, \varphi'(0) = \frac{\lambda}{\lambda-1}\mathbb{E}[\, W_\infty\,] = \frac{\lambda}{\lambda-1}, \tag{2.9}
$$

where we have used $\mathbb{E}[\, W_\infty\,] = 1$ (see [2, Theorem I.6.2]). Then, upon calculating $\varphi^{(n)}(0)$, $n \in \mathbb{N}$ recursively according to (2.8), it turns out that we can write

$$
\varphi^{(n)}(0) = K_n\, (-\rho)^{n-1}, \qquad n \in \mathbb{N} \tag{2.10}
$$

for appropriate functions $K_n = K_n(\lambda)$. For example,

$$
K_1 = K_2 = 1 \qquad \text{and} \qquad K_3 = \frac{\lambda+2}{\lambda+1}. \tag{2.11}
$$

Applying (2.10) to the right hand side of (2.8), we have

$$
\begin{aligned}
\varphi^{(n)}(0) &= \frac{1}{1-\lambda^{-(n-1)}} \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)}\, K_{n-l}(\lambda)(-\rho)^{n-l-1} K_l(\lambda)(-\rho)^{l-1} \\
&= \rho\, \Bigg[\, \frac{1}{1+\frac{1}{\lambda}+\cdots+(\frac{1}{\lambda})^{n-2}} \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)}\, K_{n-l}(\lambda) K_l(\lambda)\, \Bigg]\, (-\rho)^{n-2} \\
&= -K_n\, (-\rho)^{n-1}.
\end{aligned}
\tag{2.12}
$$

[1] We are presently *assuming* that the McLaurin expansion is valid on an interval of certain radius of convergence. This seems to be valid from simulation an we are to prove it later.

Thus, for the evaluation of $\varphi^{(n)}(0)$, the evaluation of K_n is necessary.

We write $1+\frac{1}{\lambda}+\cdots+(\frac{1}{\lambda})^{n-2} \triangleq \rho_{n-1}$ below. We note that $\rho_n \longrightarrow \rho$. Through the recursive relation

$$K_n = \frac{1}{\rho_{n-1}} \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)}\, K_{n-l}(\lambda) K_l(\lambda)$$

$$= \frac{(n-1)!}{\rho_{n-1}} \sum_{l=1}^{n-1} \lambda^{-(n-1-l)}\, \frac{K_{n-l}(\lambda)}{(n-l)!} \cdot \frac{K_l(\lambda)}{l!} \cdot (n-l), \tag{2.13}$$

we will find a simple approximation function of K_n as follows:

$$K_l = 1 + \big[\,\lambda(\lambda+1)\,\big]^{-(al+b)} l! + \varepsilon_l, \qquad\qquad l \leqq n-1. \tag{2.14}$$

We then show that the model equation (2.14) actually holds for the resulting K_n as well, below. First, applying (2.14), we can write

$$\frac{K_{n-l}}{(n-l)!} \cdot \frac{K_l}{l!} = \frac{1+(2\lambda)^{-a(n-l)-b}(n-l)!+\varepsilon}{(n-l)!} \cdot \frac{1+(2\lambda)^{-(al+b)} l! + \varepsilon}{l!}$$

$$= \frac{1+\varepsilon}{(n-l)!} \cdot \frac{1+\varepsilon}{l!} + \frac{1+\varepsilon}{l!} \big[\,\lambda(\lambda+1)\,\big]^{-a(n-l)-b}$$

$$+ \frac{1+\varepsilon}{(n-l)!} \big[\,\lambda(\lambda+1)\,\big]^{-(al+b)} + \big[\,\lambda(\lambda+1)\,\big]^{-an-2b}$$

$$\triangleq L^{(1)}_{n,l} + L^{(2)}_{n,l} + L^{(3)}_{n,l} + L^{(4)}_{n,l}.$$

Inserting these four terms into (2.13) and writing $L^{(i)}_n = \sum_{l=1}^{n-1} \lambda^{-(n-1-l)}\, L^{(i)}_{n,l} \cdot (n-l)$, we have

$$L^{(1)}_n = (1+\varepsilon)^2 \cdot \frac{1}{\rho_{n-1}} \sum_{l=1}^{n-1} \binom{n-1}{l} \lambda^{-(n-1-l)} \simeq (1+\varepsilon)^2 \cdot \frac{1}{\rho} \left(1+\frac{1}{\lambda}\right)^{n-1},$$

$$L^{(2)}_n = (1+\varepsilon) \cdot \frac{(n-1)!}{\rho_{n-1}} \sum_{l=1}^{n-1} \lambda^{-(n-1-l)}\, \frac{\big[\,\lambda(\lambda+1)\,\big]^{-a(n-l)-b}}{l!} \cdot (n-l)$$

$$\simeq (1+\varepsilon) \cdot C\, \frac{1}{\rho} \big[\,\lambda(\lambda+1)\,\big]^{-an-b}\, n!\,,$$

$$L_n^{(3)} = (1+\varepsilon) \cdot \frac{(n-1)!}{\rho_{n-1}} \sum_{l=1}^{n-1} \lambda^{-(n-1-l)} \, \frac{\big[\,\lambda(\lambda+1)\,\big]^{-al-b}}{(n-l)!} \cdot (n-l)$$

$$\simeq (1+\varepsilon) \cdot C \, \frac{1}{\rho} \, \big[\,\lambda(\lambda+1)\,\big]^{-an-b} \, (n-1)!,$$

$$L_n^{(4)} = (1+\varepsilon) \cdot \frac{(n-1)!}{\rho_{n-1}} \sum_{l=1}^{n-1} \lambda^{-(n-1-l)} \, \big[\,\lambda(\lambda+1)\,\big]^{-an-2b} \cdot (n-l)$$

$$\simeq \rho \, \big[\,\lambda(\lambda+1)\,\big]^{-an-2b} \, (n-1)!.$$

Therefore, it turns out that the recursive expression is a device that reproduces K_n with asymptotically the same form as (2.14):

$$K_n = 1 + \big[\,\lambda(\lambda+1)\,\big]^{-(an+b)} n! \cdot \big(1 + O(n^{-1})\big), \qquad \text{as} \quad n \to \infty. \tag{2.15}$$

Applying the expression (2.15) to (2.12), we have

$$\varphi(u) = \sum_{n \in \mathbb{N}_0} \frac{\varphi^{(n)}(0)}{n!} u^n = \frac{1}{\rho} \sum_{n \in \mathbb{N}_0} \frac{K_n}{n!} (-\rho u)^n$$

$$\simeq \frac{1}{\rho} \sum_{n \in \mathbb{N}_0} \frac{1 + \big[\,\lambda(\lambda+1)\,\big]^{-(an+b)} n!}{n!} \, (-\rho u)^n \; ;$$

Especially, the main term results in

$$\frac{1}{\rho} \big[\,\lambda(\lambda+1)\,\big]^{-b} \sum_{n \in \mathbb{N}_0} \Big(-\rho \big[\,\lambda(\lambda+1)\,\big]^{-a} u \Big)^n$$

$$= \frac{1}{\rho^2} \big[\,\lambda(\lambda+1)\,\big]^{a-b} \cdot \frac{1}{u + \rho^{-1} \big[\,\lambda(\lambda+1)\,\big]^{a}}.$$

Therefore, by the inverse Laplace transform, $w(x)$ has the main component of the probability density function given by

$$\frac{1}{\rho^2} \big[\,\lambda(\lambda+1)\,\big]^{a-b} \exp\big(-\big[\,\lambda(\lambda+1)\,\big]^{a} \, u \big).$$

We use the following propositions to evaluate K_n:

Proposition 1. *The functions $K_{n-l}(\lambda)K_l(\lambda)$ are non-decreasing with respect to $l = 1, \cdots, \lfloor n/2 \rfloor$:*

$$K_{n-1}(\lambda)K_1(\lambda) \geqq K_{n-2}(\lambda)K_2(\lambda) \geqq \cdots \geqq K_{\lceil n/2 \rceil}(\lambda)K_{\lfloor n/2 \rfloor}(\lambda) \qquad \textit{for} \quad \lambda \geqq 1, \tag{2.16}$$

for each $n = 3, 4, \cdots$. Especially,

$$K_{n-l}(\lambda)K_l(\lambda) \leqq K_{n-1}(\lambda)K_1(\lambda), \qquad \textit{for} \quad \lambda \geqq 1. \tag{2.17}$$

In Figure 1, $\{K_{n-l}(\lambda)K_l(\lambda)\}$ are plotted for $n = 9$ and 10. It presents that (2.16) is indeed true.

Proposition 2.

$$1 + \frac{(n-2)(n-1)}{2(\lambda+1)} \leqq K_n(\lambda) \leqq 1 + \frac{2^{-(n-1)}n!}{\lambda+1} \tag{2.18}$$

Figures. Some related graphics are plotted in here.

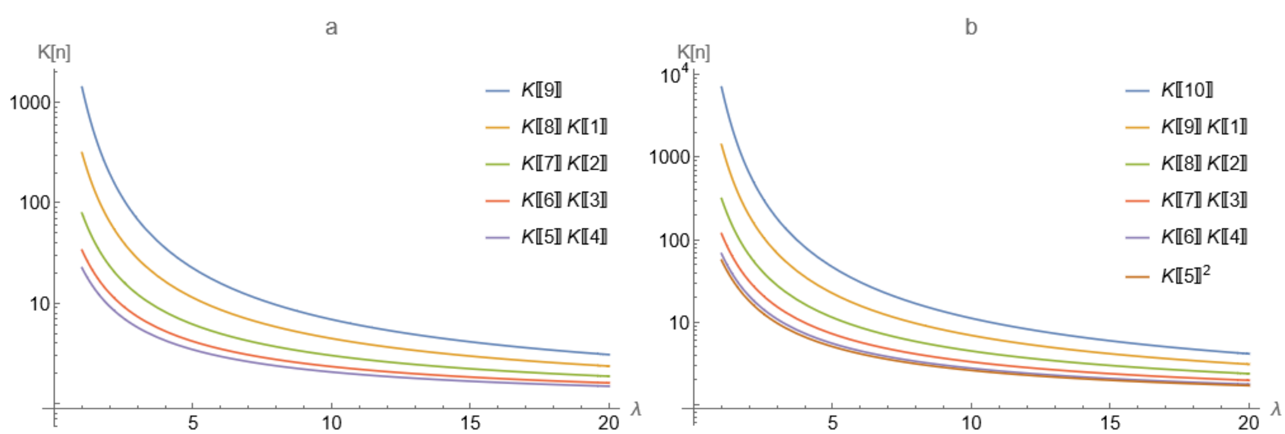

Figure 1: The graphs of $K_{n-l}(\lambda)K_l(\lambda)$ for 1: a. $n = 9$, b. $n = 10$.

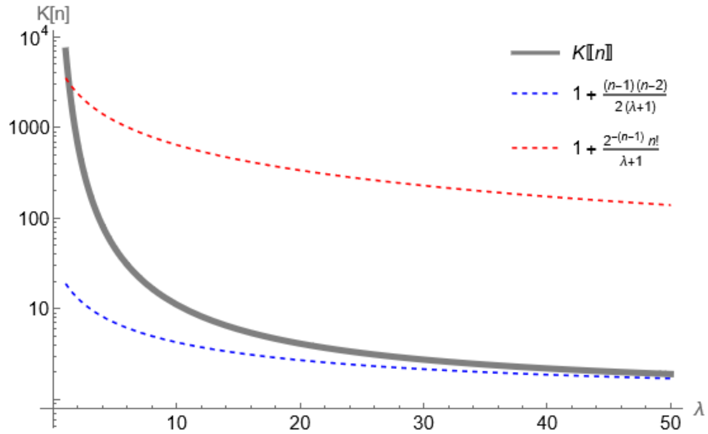

Figure 2: A graph of an upper and lower limit of the function $K_{10}(\lambda)$.

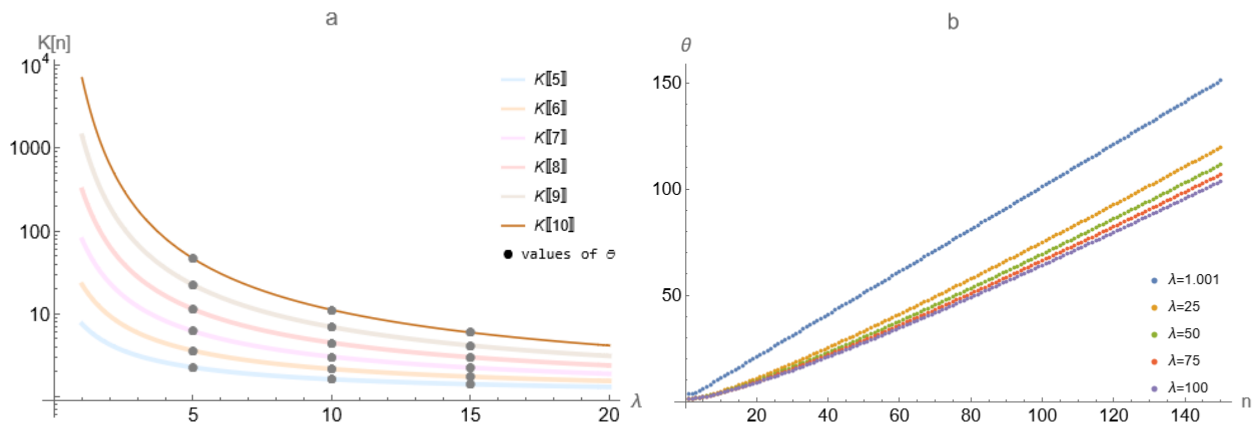

Figure 3: a. The function $K_n(\lambda)$, its relation and values of an equation $\theta = an + b$. b. Asymptotic linearity of $\theta = an + b$ in some fixed λ.

Appendix. Here, we list some of K_n. The larger the n, the more complicated the expressions become soon.

$$K_1(\lambda) = K_2(\lambda) = 1,$$

$$K_3(\lambda) = \frac{\lambda + 2}{\lambda + 1}$$

$$K_4(\lambda) = \frac{\lambda^3 + 5\lambda^2 + 6\lambda + 6}{(\lambda + 1)(\lambda^2 + \lambda + 1)}$$

$$K_5(\lambda) = \frac{\lambda^6 + 9\lambda^5 + 24\lambda^4 + 40\lambda^3 + 46\lambda^2 + 36\lambda + 24}{(\lambda + 1)^2(\lambda^2 + 1)(\lambda^2 + \lambda + 1)}$$

$$K_6(\lambda) = \frac{\lambda^{10} + 14\lambda^9 + 64\lambda^8 + 160\lambda^7 + 301\lambda^6 + 416\lambda^5 + 514\lambda^4 + 480\lambda^3 + 390\lambda^2 + 240\lambda + 120}{(\lambda + 1)^2(\lambda^2 + 1)(\lambda^2 + \lambda + 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)}$$

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