二標本ノンパラメトリック検定の連続化と局所漸近検出力

(Smoothed two-sample nonparametric tests and their local asymptotic powers)

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Abstract

We propose the smoothed median and the Wilcoxon's rank sum test. As is pointed out by Maesono et al.(2016), some nonparametric discrete tests have a problem with their significance probability. We show that the median and Wilcoxon's test can make distorted results because of this problem too. Then we propose smoothed tests of them and show these tests can solve it. The smoothed tests inherit their good properties, and we study significance probabilities and local asymptotic powers of the proposed tests.

1. Introduction

Let $X_1, X_2, \cdots, X_m$ be independently and identically distributed random variables ($i.i.d.$) from distribution function $F(x)$ and $Y_1, Y_2, \cdots, Y_n$ be $i.i.d.$ from $F(x-\theta)$ where θ is unknown location parameter. We assume that $m+n=N$ and $0 < \lim_N m/N = \lim_N \lambda_N = \lambda < 1$. We consider '2-sample problem' whose null hypothesis $H_0 : \theta = 0$ and alternative $H_1 : \theta > 0$. There are many nonparametric tests based on linear order statistic (see Hájek et al.(1999)). The median and the Wilcoxon's rank sum test are widely used and investigated well among them. Moreover, the median test is known for its low cost especially in survival analysis because it only needs the number of the lower values than the combined median. However, Freidlin & Gastwirth(2000) have reported that the median test is inferior even in double exponential case in spite of its theoretical most powerfulness. We first investigate the median's power again and find theoretical and numerical superiority in case the distribution has heavy tail.

As is shown by Maesono et al.(2016), the sign test and the Wilcoxon's signed rank test have a problem of their significance probabilities. It comes from their discreteness of p-values. Because of this, we can make an arbitrary statistical decision, and we show that the median and the Wilcoxon's rank sum test also suffer from this problem. Using a smooth function in the same way as Maesono et al.(2016), we propose new smoothed test statistics. It is proved that their asymptotic properties including Pitman's $A.R.E.$ are the same respectively. Moreover, we show that the smoothed median test inherits the cost-effectiveness and more locally asymptotically powerful both theoretically and numerically.

Keywords: nonparametric test, kernel estimator, significance probability, local asymptotic power.

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2. Median and Wilcoxon's rank sum test

2.1. Median and Wilcoxon's rank sum test

In this section, we introduce the median and the Wilcoxon's rank sum test statistic and their properties. Some numerical experiments on their power are studied and the problem about their p-values are brought up here.

Let us define that $\psi(x) = 1 \ (x \geq 0), \quad = 0 \ (x < 0)$ and $Z_1 = X_1, \cdots, Z_m = X_m, \cdots, Z_{m+1} = Y_1, \cdots, Z_N = Y_n$. There are various forms of the median test statistic, and here we define

$$M = M(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{j=1}^{n} \psi(Y_j - Z)$$

where $\boldsymbol{X} = (X_1, X_2, \cdots, X_m)^T$ and Z denotes the sample median of $\{Z_1, Z_2, \cdots, Z_N\}$. We put $Z = Z_{((N+1)/2)}$ if N is odd and $Z = (Z_{(N/2)} + Z_{((N/2)+1)})/2$ else where $Z_{(1)} < \cdots < Z_{(N)}$ are the order statistics. The Wilcoxon's rank sum test statistic is given by

$$W_2 = W_2(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} \psi(Y_j - X_i).$$

For observed values $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)^T$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^T$, we put $m = m(\boldsymbol{x}, \boldsymbol{y})$ and $w_2 = w_2(\boldsymbol{x}, \boldsymbol{y})$ which are the realized values of M and W_2. If the p-value

$$P_0(M \geq m) \quad \text{or} \quad P_0(W_2 \geq w_2)$$

is small enough, we reject the null hypothesis H_0. Here $P_0(\cdot)$ denotes a probability under the null hypothesis H_0.

As we can see easily, the median test only needs the number of the lower values than the combined median. In survival data analysis, we can finish the observation whenever the combined median is obtained. If the number of data is large or the tail of the distribution is heavy, it is possible to save much cost and time. However, not many statisticians take the great merit into account. We show the smoothed median test inherits the superiority in Section 3.1.

2.2. Local power

The median and the Wilcoxon's rank sum test are one of linear rank tests which are defined as follows

$$S = S(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{j=1}^{n} a_{m,n}(R_j)$$

where $R_j (j = 1, \cdots, n)$ denotes rank of the observation Y_j in the combined sample $Z_1, \cdots, Z_N$. The Wilcoxon's test is given by

$$a_{m,n}(u) = m + \frac{1}{2}(n+1) - u,$$

so the test statistic is

$$W_2 = W_2(\boldsymbol{X}, \boldsymbol{Y}) = mn + \frac{1}{2}n(n+1) - \sum_{j=1}^{n} R_j.$$

The last term

$$U = \sum_{j=1}^{n} R_j$$

Table 1: Pitman's $A.R.E.$ of 2-sample tests

population distribution	Nor.	Logis.	D.exp.
$ARE(M\|T_2)$	0.637	0.822	2
$ARE(W_2\|T_2)$	0.955	1.10	1.5
$ARE(M\|W_2)$	0.667	0.75	1.33

Table 2: Pitman's $A.R.E.$ of 2-sample tests in T distribution

population distribution	$T(2)$	$T(1)$	$T(1/2)$
$ARE(M\|W_2)$	0.961	1.33	2.29

is led as locally most powerful rank test in logistic case, and W_2 is equivalent to the test statistic U (Hájek et al. (1999)). The median test statistic is obtained by

$$a_{m,n}(u) = \psi(u - \frac{1}{2}(m+n+1)),$$

and so

$$M = M(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{j=1}^{n} \psi(R_j - \frac{1}{2}(m+n+1)).$$

M is asymptotically equivalent to the locally most powerful rank test statistic when the underlying distribution is the double exponential.

Table 1 shows Pitman's asymptotic relative efficiencies ($A.R.E.$) of the combinations of the two-sample t-test T_2, the median and the Wilcoxon's test. Pitman's $A.R.E.$s are given by a ratio of asymptotic local power of two tests. We confirm that T_2 is the most powerful in normal case, M is in double exponential case and so on. Note that $ARE(M|W_2) = 1.33$ in the double exponential case.

Nevertheless, the numerical weakness of the median test's power was reported. Freidlin & Gastwirth (2000) show the empirical local power of some two-sample linear rank tests, and in their tables, the median test is inferior to Wilcoxon's test even in the double exponential case.

We investigate their powers both theoretically and numerically in heavy tailed case. Table 2 shows Pitman's $ARE(M|W_2)$ of $T(2)$ (T distribution with 2 degrees of freedom), $T(1)$ (Cauchy distribution) and $T(1/2)$. Distribution's tail gets heavy in accordance with decrease in the degree and we can prove the monotonic increase of $ARE(M|W_2)$.

Table 3 and 4 are the numerical results of the ratio of their empirical local power in the t distributions. We find that the power of the median test is so stronger especially in $T(1/2)$. If to obtain complete data needs much cost and time in such case, the median test has great superiority.

2.3. Significance probability

We will show that the median test and the Wilcoxon's rank sum test have a problem with their significance probabilities. Maesono et al.(2016) report that the sign and the Wilcoxon's signed rank test can make distorted results and that this problem comes from their p-values' discreteness. The median and the Wilcoxon's test are also discrete, and we study their significance probabilities.

Table 3: Ratio of empirical local power in $T(1)$ distribution

sample size (m, n)	(10,10)	(20,20)	(30,30)	(50,50)	(10,30)	(30,10)
$\theta = 1$	1.03	1.16	1.17	1.09	1.06	1.06
$\theta = 0.5$	1.01	1.15	1.17	1.16	1.07	1.06
$\theta = 0.1$	0.994	1.03	1.05	1.05	1.04	0.993

Table 4: Ratio of empirical local power in $T(1/2)$ distribution

sample size (m, n)	(10,10)	(20,20)	(30,30)	(50,50)	(10,30)	(30,10)
$\theta = 1$	1.26	1.52	1.54	1.54	1.34	1.34
$\theta = 0.5$	1.17	1.37	1.42	1.57	1.27	1.26
$\theta = 0.1$	1.06	1.08	1.10	1.14	1.07	1.06

Table 5 shows the ratio of frequency of exact p-value of W_2 getting smaller than M in the tale area Ω_α

$$\Omega_\alpha = \left\{ \boldsymbol{x} \in \boldsymbol{R}^n \;\middle|\; \frac{m(\boldsymbol{x}) - E_0(M)}{\sqrt{V_0(M)}} \geq v_{1-\alpha}, \quad \text{or} \quad \frac{w_2(\boldsymbol{x}) - E_0(W_2)}{\sqrt{V_0(W_2)}} \geq v_{1-\alpha} \right\}$$

where $v_{1-\alpha}$ is a $(1-\alpha)th$ quantile of the standard normal distribution $N(0,1)$, and $E_0(\cdot)$ and $V_0(\cdot)$ stand for an expectation and a variance under H_0, respectively. We count samples that an exact p-value of the test is smaller than the other in the tail area Ω_α, and calculate the ratio of the frequency.

Because the values in Table 5 are larger than 1, we find that W_2 tends to have smaller p-value than M. They can let one to use W_2 if one wants the small p-value. This comes from that the possible p-values of M are more sparse than W_2 like the sign and Wilcoxon's signed rank test as is discussed by Maesono et al.(2016).

In order to conquer the problem, we propose the smoothed median test $\widetilde{M}$ and the Wilcoxon's test $\widetilde{W}_2$ and investigate their asymptotic properties. The discrete tests are distribution-free, but the smoothed tests are not. However, we can confirm that they are asymptotically distribution-free and their Pitman's $A.R.E.$'s are exactly same to M and W_2.

Table 5: Comparison of significance probabilities

sample size (m, n)	(10,10)	(20,20)	(30,30)	(10,20)	(20,10)	(U_m^*, U_n^*)[1]
$z_{0.9}$	2.21	2.79	1.68	7.13	7.07	3.86
$z_{0.95}$	6.37	5.66	2.17	3.44	3.41	4.21
$z_{0.975}$	3.05	2.80	3.72	16.6	14.7	4.10
$z_{0.99}$	33.7	7.54	1.56	6.05	5.29	4.11

[1] U_m^* and U_n^* are random numbers from the discrete uniform distribution $U^*(5, 40)$

3. Smoothed median test

3.1. Smoothed median test

From now on, we assume that N is odd for the brevity, and we consider to make M smooth appropriately. A possible way is to use kernel type statistics. The empirical distribution function of a population distribution function F is given by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n}\psi(x - X_i).$$

Let $k(u)$ be a kernel function which satisfies

$$\int_{-\infty}^{\infty} k(t)dt = 1,$$

and $K(t)$ is an integral of $k(t)$

$$K(t) = \int_{-\infty}^{t} k(u)du.$$

In this paper, we assume that the kernel k is a symmetric function around the origin. The smooth kernel estimator of F is given by

$$\widehat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} K\Big(\frac{x - X_i}{h}\Big)$$

where h is a bandwidth which satisfies $h \to 0$ and $nh \to \infty$. Since

$$M = \frac{n - m + 1}{2} - \sum_{i=1}^{m}\psi^*(Z - X_i)$$

where $\psi^*(x) = 1 \ (x > 0), \quad = 0 \ (x \le 0)$, and we set $M^\dagger$ as the second term. Applying the kernel method to $M^\dagger$, we define the following smoothed test statistic

$$\widetilde{M} = \sum_{i=1}^{m} K^*\left(\frac{Z - X_i}{h}\right)$$

where K^* is an integral of the kernel $k^*(t)$ which satisfies

$$\int_{0}^{\infty} k^*(t)dt = 1 \quad \text{and} \quad k^*(t) = 0 \ \text{ for } \ t \le 0$$

and h is a bandwidth. In addition, we assume that $A^*_{1,1} = 0$ where

$$A^*_{i,j} = \int_{-\infty}^{\infty} t^i k^{*j}(t)dt.$$

As we see easily from the definition, the smoothed median test does not need the values of the data larger than the combined median. Therefore, $\widetilde{M}$ inherits the cost-effectiveness.

Remark 2 The above condition of k^* is not tough, and we can easily construct the following simple polynomial type one

$$k^*(t) = (-6t + 4)\, I(0 < u < 1).$$

It is easy to construct a kernel function which satisfies $A^*_{1,1} = A^*_{2,1} = 0$ too. It is not a problem that k^* may take negative value, because $\widetilde{M}$ is test statistic.

3.2. Asymptotic properties

The median test statistic $M^\dagger$ exactly follows the hypergeometric distribution $HG(N, m, (N-1)/2)$ under H_0, and so we easily find that

$$E_0[M^\dagger] = \frac{m}{2}\left(1 - \frac{1}{m+n}\right), \quad V_0[M^\dagger] = \frac{mn}{4(m+n)}.$$

Using the following joint distribution of Z and $U(\in \{1, \cdots, r\})$ which is the number of $\{X_1, \cdots, X_m\}$ less than Z given by

$$\begin{aligned} & h(u,z) \\ = \ & m\binom{m-1}{u}\binom{n}{r-u}F^u(z)[1-F(z)]^{m-u-1}F^{r-u}(z-\theta)[1-F(z-\theta)]^{n-r+u}f(z) \\ & +n\binom{m}{u}\binom{n-1}{r-u}F^u(z)[1-F(z)]^{m-u}F^{r-u}(z-\theta)[1-F(z-\theta)]^{n-r+u-1}f(z-\theta) \end{aligned}$$

under H_1. Mood(1954) obtains the following asymptotic expectation

$$E_\theta[M^\dagger] = mF(z_{\theta,N}) + o(N)$$

where $z_{\theta,N}$ stands for the median of the distribution $G_{\theta,N}$ defined as

$$G_{\theta,N}(x) = \lambda_N F(x) + (1-\lambda_N)F(x-\theta).$$

Then we find the following Pitman efficiency

$$e_P[M^\dagger] = \lim_N \left[(NV_0[M^\dagger])^{-1/2} \frac{\partial}{\partial\theta} E_\theta[M^\dagger] \bigg|_{\theta=0} \right] = 2\sqrt{\lambda(1-\lambda)}f(z_0).$$

Using the Bahadur representation (Bahadur(1966)) of Z and the asymptotic expansion under H_0 and $f(z_0) > 0$ where z_0 is the median of F, we have

$$E_0[\widetilde{M}] = \frac{m}{2}\left\{1 - \frac{1}{m+n} + O(h^2)\right\} + o(N^{1/2}).$$

In the same manner, we have

$$V_0[\widetilde{M}] = \frac{mn}{4(m+n)} + o(N).$$

Using the Bahadur representation for quantiles of two samples (Liu & Yin(1994)) under H_1, and both $f(z_{\theta,N})$ and $f(z_{\theta,N} - \theta)$ are positive, we have

$$\widetilde{M} = \sum_{i=1}^m \left[K^*\left(\frac{z_{\theta,N} - X_i}{h}\right) + \frac{1}{g_{\theta,N}(z_{\theta,N})h} k^*\left(\frac{z_{\theta,N} - X_i}{h}\right)\left\{\frac{1}{2} - G_{\theta,(N)}(z_{\theta,n})\right\}\right] + o_P(n^{1/2})$$

where

$$G_{\theta,(N)}(x) = \lambda_N F_{X,(m)}(x) + (1-\lambda_N)F_{Y,(n)}(x)$$

and $F_{X,(m)}$, $F_{Y,(n)}$ are the empirical distribution function of $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_n\}$ respectively. Therefore, we obtain the following asymptotic expectation under H_1

$$\begin{aligned} E_\theta[\widetilde{M}] &= m\int_{-\infty}^{\infty} k^*(v)\, F(z_{\theta,N} - hv)dy + o(N^{1/2}) \\ &= mF(z_{\theta,N}) + o(N^{1/2}). \end{aligned}$$

Combining the above results, we can prove that the Pitman efficiency is same as the discrete one :

$$e_P[\widetilde{M}] = e_P[M^\dagger]$$

Since the main term of $\widetilde{M}$ is a two-sample U statistic, it is easy to prove its asymptotic normality. Note that the main term of the asymptotic expectation and variance of $\widetilde{M}$ do not depend on F under H_0. Moreover, we can evaluate the residual using the Edgeworth expansion.

Theorem 3.1 *Let us assume that f' exists and is continuous at a neighborhood of z_0, and $f(z_0) > 0$. Furthermore, we set that $A^*_{1,1} = 0$ and $h = o(N^{-1/4})$ or that $A^*_{1,1} = A^*_{1,2} = 0$ and $h = o(N^{-1/6})$. Then, we have*

$$\begin{aligned} & \sup_{-\infty<x<\infty} \left| P_0 \left[V_1^{-1/2}(\widetilde{M} - E_1) < x \right] - \Phi(x) \right| \\ = \; & O(N^{-1/4} + A^*_{1,j}\sqrt{N}h^j) \end{aligned}$$

where

$$E_1 = \frac{m}{2}\left(1 - \frac{1}{m+n}\right), \quad V_1 = \frac{mn}{4(m+n)}$$

and Φ is the standard normal distribution function.

3.3. Local power

Here we study the local power of $\widetilde{M}$. Put $\theta = N^{-1/2}\xi$ and $z_{\theta,N} = z_0 + N^{-1/2}\eta + o(N^{-1/2}h)$ where η is a constant number and z_0 is defined before. Since the following expansion holds

$$\begin{aligned} \frac{N}{2} &= mF(z_{\theta,N}) + nF(z_{\theta,N} - \theta) \\ &= NF(z_{\theta,N}) - \frac{n}{\sqrt{N}} f(z_{\theta,N})\xi + O(N^{-1}), \end{aligned}$$

we have

$$\sqrt{N} f(z_0)\eta = \frac{n}{\sqrt{N}} f(z_0)\xi + o(N^{1/2}h).$$

Then, we find $z_{\theta,N} = z_0 + N^{-1/2}(1-\lambda)\xi + o(N^{-1/2}h)$.

Combining

$$V_0[\widetilde{M}] = \frac{mn}{4(m+n)} - 2(mh)A^*_{1,1,1} f(z_0) + O\left(Nh^2 + \frac{\log N}{\sqrt{N}}\right)$$

where

$$A^*_{i,j,l} = \int_{-\infty}^{\infty} t^i k^{*j}(t) K^{*l}(t) dt$$

and

$$\begin{aligned} E_\theta[\widetilde{M}] - E_0[\widetilde{M}] &= mF(z_{\theta,N}) - \frac{m}{2} + O(Nh^2) \\ &= \lambda(1-\lambda)\sqrt{N} f(z_0)\xi + o(N^{1/2}h), \end{aligned}$$

we find the local power of $\widetilde{M}$ is given by

$$LP_{\frac{\xi}{\sqrt{N}},\alpha}[\widetilde{M}] = P_{\frac{\xi}{\sqrt{N}}}\left[V_1^{\star-1/2}(\widetilde{M}-E_1) > v_{1-\alpha}\right] = 1-\Phi(v_{1-\alpha}-c)+O(N^{-1/2}+h^2)$$

where

$$V_1^{\star} = V_1 - 2(mh)A^*_{1,1,1}f(z_0),$$

$$c = \xi\left[2\sqrt{\lambda(1-\lambda)}f(z_0)+8h\sqrt{\frac{\lambda}{1-\lambda}}A^*_{1,1,1}f^2(z_0)\right]$$

and $v_{1-\alpha}$ is the $(1-\alpha)th$ quantile of the standard normal distribution. $M^{\dagger}$ doesn't have the second term of c of order h, so we can easily find the following inequality.

Theorem 3.2 *Under the same assumptions of Theorem 3.1 and that* $\sqrt{N}h \to \infty$*, if* $A^*_{1,1,1}$ *is positive, we have*

$$\lim_{N\to\infty}\frac{1}{h}(LP_{\frac{\xi}{\sqrt{N}},\alpha}[\widetilde{M}] - LP_{\frac{\xi}{\sqrt{N}},\alpha}[M^{\dagger}]) > 0.$$

We can construct the following polynomial type kernel

$$k^*(t) = \left[\left(\frac{\pm 3\sqrt{4353}-3}{17}\right)u^2 + \left(\frac{\mp 3\sqrt{4353}-99}{17}\right)u + \left(\frac{\pm\sqrt{4353}+135}{34}\right)\right]I(0<u<1).$$

which satisfies $A^*_{1,1}=0$ and $A^*_{1,1,1}=1(>0)$. In the sense of the local power, the optimal bandwidth h under the assumption is like $h=O(N^{-1/4}/\log N)$.

4. Smoothed Wilcoxon's rank sum test

4.1. Smoothed Wilcoxon's rank sum test

Here, we give the smoothed test $\widetilde{W}_2$ in the same manner. We define the smoothed test statistics

$$\widetilde{W}_2 = \sum_{i=1}^{m}\sum_{j=1}^{n}K\left(\frac{Y_j-X_i}{h}\right),$$

and we assume $A_{1,1}=0$.

4.2. Asymptotic properties

The following moments of Wilcoxon's test statistic W_2 are well known

$$E_\theta[W_2] = mn\int_{-\infty}^{\infty}f(y)F(y+\theta)dy, \quad V_0[W_2] = \frac{mn(m+n)}{12},$$

and then we get the following Pitman efficiency

$$e_P[W_2] = \sqrt{12\lambda(1-\lambda)}\int f^2(x)dx.$$

Using variable changes and the Taylor expansion, we obtain the asymptotic expectation of $\widetilde{W}_2$

$$\begin{aligned} E_\theta[\widetilde{W}_2] &= mn\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K\left(\frac{x-y}{h}\right)f(x-\theta)f(y)dxdy \\ &= mn\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k(v)f(y+hv-\theta)F(y)dvdy \\ &= mn\left[\int_{-\infty}^{\infty}f(y)F(y+\theta)dy+O(h^2)\right]. \end{aligned}$$

Hereafter, we assume that $h = o(N^{-1/2})$ or that $A_{1,1} = \cdots = A_{3,1} = 0$ and $h = o(N^{-1/4})$ in order to ignore the residual term. It is easy to obtain such kernels using Jones & Signorini (1997).

Further, it is easy to see that $V_0[\widetilde{W}_2] = mn(m+n)/12 + o(N^3)$. Combining the above results, we find that the Pitman efficiency is same

$$e_P[\widetilde{W}_2] = e_P[W_2].$$

Using the asymptotic theory for two-sample U-statistics, we can prove the asymptotic normality and the following validity of the Edgeworth-type expansion.

Theorem 4.1 *Let us assume that $h = o(N^{-1/2})$ or that $A_{1,2} = 0$ and $h = o(N^{-1/4})$. Then, we have*

$$\begin{aligned} &\sup_{-\infty<x<\infty} \left| P_0 \left[V_2^{-1/2}(\widetilde{W}_2 - E_2) < x \right] - \Phi(x) \right| \\ = \ & o(N^{-1/2}) \end{aligned}$$

where

$$E_2 = \frac{mn}{2}, \quad V_2 = \frac{mn(m+n)}{12}.$$

4.3. Local power

In the same manner as section 3.3, we can obtain the local asymptotic power of $\widetilde{W}_2$.

Theorem 4.2 *Under the same assumptions of Theorem 4.1 and that $\sqrt{N}h \to \infty$, we have*

$$\lim_{N\to\infty} \frac{1}{h}(LP_{\frac{\xi}{\sqrt{N}},\alpha}[\widetilde{W}_2] - LP_{\frac{\xi}{\sqrt{N}},\alpha}[W_2]) = 0$$

where

$$LP_{\frac{\xi}{\sqrt{N}},\alpha}[\widetilde{W}_2] = P_{\frac{\xi}{\sqrt{N}}} \left[V_2^{-1/2}(\widetilde{W}_2 - E_2) > v_{1-\alpha} \right].$$

5. Simulation study

In this section, we compare the significance probabilities of $\widetilde{M}$ and $\widetilde{W}_2$ by simulation because the distributions of $\widetilde{M}$ and $\widetilde{W}_2$ depend on F. For 100,000 times random samples from the standard normal distribution, we estimate the significance probabilities in the tale area

$$\widetilde{\Omega}_\alpha = \left\{ \boldsymbol{x} \in \boldsymbol{R}^n \;\middle|\; \frac{\widetilde{m}(\boldsymbol{x}) - E_0(\widetilde{M})}{\sqrt{V_0(\widetilde{M})}} \geq v_{1-\alpha}, \quad \text{or} \quad \frac{\widetilde{w}_2(\boldsymbol{x}) - E_0(\widetilde{W}_2)}{\sqrt{V_0(\widetilde{W}_2)}} \geq v_{1-\alpha} \right\}.$$

Similarly as Table 5, Table 6 stands for the ratio of the samples that the significance probability of $\widetilde{W}_2$ is smaller than $\widetilde{M}$. Comparing Table 5 and 6, we can see that the differences of the p-values of $\widetilde{M}$ and $\widetilde{W}_2$ is smaller than those of S and W.

Simulation results of their powers will be shown in the talk.

Table 6: Relation of significance probabilities

sample size (m,n)	(10,10)	(20,20)	(30,30)	(10,20)	(20,10)	$(U_m^*, U_n^*)^2$
$z_{0.9}$	1.41	1.08	1.31	1.05	1.22	0.827
$z_{0.95}$	0.829	1.21	1.39	1.41	1.24	0.782
$z_{0.975}$	1.64	1.12	1.09	1.23	1.39	0.755
$z_{0.99}$	0.985	1.03	1.32	0.707	0.806	0.677

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