組み合わせ論と無限分解可能分布のひとつの接点

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Flow chart of my talk

1. Extension 1 of Bender and Canfield (1996):
   - Probabilistic Proof
     - Moment generating function of CP
     - Results of Hansen (1988)
2. Combinatorial Proof for Extension 1 of BC (1996):
   - Basics of Combinatorics
     - Cycle index polynomials (CI)
     - Generating func. and recurrence eq. of CI
   - Combinatorial Proof for Extension 1
3. Extension 2 of BC (1996):
   - Extension of Structural results in BC (1996)
4. Normal and Binomial Convolution of Sequences with BC Condition
   - Result by Schirmacher (1999) and its Extension

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Extension 1 of BC (1996)'s theorem:

$(a_k)_{k\geq 0}$, $(b_k)_{k\geq 0}$ and $(c_k)_{k\geq 1}$: non-negative sequences linked by

$$\sum_{k=0}^{\infty} a_k u^k = \sum_{k=0}^{\infty} \frac{b_k u^k}{k!} = \exp\Big(\sum_{j=1}^{\infty} \frac{c_j u^j}{j}\Big). \tag{1.1}$$

Assume that $(c_k/k)_{k\geq 0}$ are infinitely summable. Suppose that $(c_k)_{k\geq 1}$ is log-concave and $c_1^2 \geq c_2$, then

$$a_{k-1}a_{k+1} \leq a_k^2 \leq \frac{k+1}{k} a_{k-1}a_{k+1}, \tag{1.2}$$

$$b_{k-1}b_{k+1} \geq b_k^2 \geq \frac{k}{k+1} b_{k-1}b_{k+1}. \tag{1.3}$$

Moreover, the left ineq. in (1.2) or equivalently the right ineq. of (1.3) holds only if $c_1^2 \geq c_2$.

Extension 1 of BC (1996)'s theorem:

Conversely, suppose that $(c_k)_{k\geq 1}$ is log-convex, then

$$a_{k-1}a_{k+1} \geq a_k^2, \tag{1.4}$$

$$b_k^2 \leq \frac{k}{k+1} b_{k-1}b_{k+1} \tag{1.5}$$

holds if and only if $c_1^2 \leq c_2$.

- The extended part is the log-convexity results of $(a_k)_{k\geq 0}$.
- We can not recover the structural result (BC (1996) theorem 2) for (b_k) from probabilistic proof, i.e. $(n+1)b_m b_n - m b_{m-1} b_{n+1}$ for $1 \leq m \leq n$ could be expressed as a polynomial in $\mathscr{Y} = \{c_1, c_2, \ldots\} \cup \{c_j c_k - c_{j-1}c_{k+1} : 0 < j \leq k\}$ with non-negative integer coefficients.

Main idea of proof

N : Poisson r.v. $P(N=k) = \frac{\lambda^k}{k!}e^{-\lambda}$, $k \in \mathbb{Z}_+$.
$S_N := \sum_{j=1}^N X_j$: CP r.v.
$P_n = P(S_N = n)$, $n \in \mathbb{Z}_+$ and $f_n := P(X_1 = n)$.

Probability GF of S_N is for $|u| \leq 1$,

$$\sum_{k=0}^{\infty} P_k u^k = \exp\Big(\lambda\big(\sum_{j=0}^{\infty} f_j u^j - 1\big)\Big) = e^{\lambda(f_0-1)} \exp\Big(\sum_{j=1}^{n} \frac{\lambda j f_j}{j} u^j\Big). \tag{1.6}$$

Here putting $a_k = e^{-\lambda(f_0-1)}P_k$ and $c_k = \lambda k f_k$, the relation (1.1) is recovered. Then, once log-concavity (resp. log-convexity) of (P_k) is proven, that of (a_k) follows.
Notice the known relation between (P_k) and $(c_k = \lambda k f_k)$ for CP.

Results for CP : Hansen (1988)

Hansen (1988) [Theorems 1 and 2]

$$P_0 = e^{\lambda(f_0-1)}, \quad (n+1)P_{n+1} = \sum_{k=0}^{n} \lambda(k+1) f_{k+1} P_{n-k}, \qquad (1.7)$$

Let $(P_n)_{n\geq 0}$ and $(f_k)_{k\geq 0}$ be connected by (1.7). Assume $(kf_k)_{k\geq 1}$ is log-concave (resp. log-convex), then (P_n) is log-concave (resp. log-convex) if and only if $\lambda f_1^2 - 2f_2 \geq 0$ (resp. $\lambda f_1^2 - 2f_2 \leq 0$).

Proof : for convenience write $r_k = \lambda(k+1)f_{k+1}$ so that the recursion (1.7) has

$$(n+1)P_{n+1} = \sum_{k=0}^{n} r_k P_{n-k}. \qquad (1.8)$$

We are starting with two Lemmas.

Lemmas 1 and 2 of Hansen (1998)

Assume (1.8) and let $P_{-1} = 0$. Then

$$\begin{aligned} m(m+2)(P_{m+1}^2 - P_m P_{m+2}) &= P_{m+1}(r_0 P_m - P_{m+1}) \\ &+ \sum_{\ell=0}^{m}\sum_{k=0}^{\ell}(P_{m-\ell}P_{m-k-1} - P_{m-k}P_{m-\ell-1})(r_{k+1}r_\ell - r_{\ell+1}r_k), \end{aligned}$$

$$\begin{aligned} r_{m+1}(m+2)(P_{m+1}P_{m+3} - P_{m+2}^2) &= P_{m+1}(r_{m+2}P_{m+2} - r_{m+1}P_{m+3}) \\ &+ \sum_{k=0}^{m}(P_{m-k}P_{m+2} - P_{m+1}P_{m-k+1})(r_{m+2}r_k - r_{k+1}r_{m+1}). \end{aligned}$$

Assume (1.8) and $P_0 > 0$, then
(i) if (P_n) is strictly log-concave for $n = 1, 2, \ldots, m$, then
$r_0 P_m - P_{m+1} > 0$,
(ii) if (r_n) is strictly log-convex and $r_0^2 - r_1 < 0$, then
$r_{m+2}P_{m+2} - r_{m+1}P_{m+3} > 0$.

Proof for extended part

Noticing $\sum_{k=1}^{\infty}(c_k/k) < \infty$, we take some $\lambda > \sum_{k=1}^{\infty} c_k/k$ since λ could be any positive constant. Let $c_k = \lambda f_k k$, $k \geq 1$ in (1.1) and put $f_0 = 1 - \sum_{k=1}^{\infty} f_k$. Then multiply both sides by $e^{\lambda(f_0-1)}$ to obtain

$$\sum_{k=0}^{\infty} e^{\lambda(f_0-1)} a_k u^k = e^{\lambda(f_0-1)} \exp\left(\sum_{j=1}^{\infty} \frac{\lambda j f_j}{j} u^j\right).$$

Since the right-hand side is probability GF of P_k by the uniqueness $P_k = e^{\lambda(f_0-1)} a_k$ holds.

The log-concavity (resp. log-convexity) of $(c_k)_{k\geq 1}$ and that of $(kf_k)_{k\geq 1}$ are equivalent. Moreover $\lambda f_1^2 - 2f_2 \geq 1/\lambda \cdot (c_1^2 - c_2 \cdot 1) \geq 0$.

Then conclusions of Theorem are implied by

$$a_k^2 - a_{k-1}a_{k+1} = e^{-2\lambda(f_0-1)}(P_k^2 - P_{k-1}P_{k+1})$$

together with results of Theorems 1 and 2 of Hansen (1988).

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Disjoint cycle representation of permutations

Permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 4 & 7 & 1 & 5 & 3 \end{pmatrix}$$

cycles : $1 \to 6 \to 5 \to 1, \quad 2 \to 2, \quad 3 \to 4 \to 7 \to 3.$

$$\sigma = (165)(2)(347)$$

The permutation σ is the product of three cycles, one of the length 3, fixed point, one of the length 3.

Every permutation can be written as a product of disjoint cycles uniquely except for the order of cycles in the product.

Disjoint cycle representation of permutations

Cycle index monomial associated with permutation σ

$$\prod_{k=1}^{n} c_k^{j_k(\sigma)}$$

c_k : the variable correspond to length of k cycles
$j_k(\sigma)$: number of cycles of σ of length k

$$0 \leq j_k(\sigma) \leq \lfloor n/k \rfloor \quad \text{and} \quad \sum_{k=1}^{n} k j_k(\sigma) = n.$$

$\sigma = (165)(2)(347)$: cycle index monomial $c_1^1 c_3^2$.

Cycle index of a permutation group

$X = \{1, 2, \ldots, n\}$: object set, Σ_n : permutation group.
$|\Sigma_n|$: the order of Σ_n, the number of permutations in Σ_n.
the degree of Σ_n is the number n of elements in the object set X.

> Cycle index of permutation group Σ_n, denoted by $Z(\Sigma_n)$ is the average of the cycle index monomials of all the permutations σ in Σ_n.

$$Z(\Sigma_n) = |\Sigma_n|^{-1} \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} c_k^{j_k(\sigma)}.$$

(Harary and Palmer (1973), R.P. Stanley (2012))

Cycle index of a symmetric group

The CI of the sym. group Σ_n satisfies the recurrence relation (wiki)

$$Z(\Sigma_n) = n^{-1} \sum_{k=1}^{n} c_k Z(\Sigma_{n-k}).$$

Proof. Let $Z(\Sigma_0) = 1$, we pick up the cycle which contains n and has the size k, $1 \le k \le n$. There are $\binom{n-1}{k-1}$ ways to choose the remaining $k-1$ element and $k!/k$ different cycles.

$$Z(\Sigma_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{k=1}^{n} c_k^{j_k(\sigma)} = \frac{1}{n!} \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k!}{k} c_k \cdot (n-k)! Z(\Sigma_{n-k})$$
$$= n^{-1} \sum_{k=1}^{n} c_k Z(\Sigma_{n-k}).$$

Generating func. of CI

Type $\mathbf{j} = (j_1, \ldots, j_n)$ s.t. $\sum_k k j_k = n$. Then the number of permutations of n letters that have $\mathbf{j}$ for their cycle type is

$$\frac{n!}{\prod_{k \ge 1} (k^{j_k} j_k!)}, \qquad \text{e.g. Wilf (1994), Stanley (2012).}$$

Hence

$$Z(\Sigma_n) = \sum_{j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n} \frac{1}{\prod_{k=1}^{n} k^{j_k} j_k!} \prod_{k=1}^{n} c_k^{j_k}$$

from which we get

$$\sum_{n \ge 0} Z(\Sigma_n) u^n = \sum_{j_1 \ge 0} \frac{(uc_1)^{j_1}}{1^{j_1} j_1!} \cdot \sum_{j_2 \ge 0} \frac{(uc_2)^{j_2}}{1^{j_2} j_2!} \cdots$$
$$= \exp\left(c_1 u + c_2 \frac{u^2}{2} + c_3 \frac{u^3}{3} + \cdots\right).$$

Combinatorial proof for extended part

Σ_m : symmetric group of degree m.
$i_j(\sigma)$: number of j-cycle in the permutation σ.
Then the CI polynomials (a_m) and related polynomials (b_m) are

$$a_m(c_1, c_2, \ldots, c_m) = \frac{1}{m!} b_m(c_1, c_2, \ldots, c_m) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} wt(\sigma),$$

where $wt(\sigma) = c_1^{i_1(\sigma)} \cdots c_m^{i_m(\sigma)}$.

Properties : For $\sigma_1 \in \Sigma_{m+1}$ let σ_1' be σ_1 with $m+1$th element deleted from the cycle containing it. The summation of $wt(\sigma_1')$ over all $\sigma_1 \in \Sigma_{m+1}$ yields

$$\sum_{\sigma_1 \in \Sigma_{m+1}} wt(\sigma_1') = (m+1) b_m. \tag{2.9}$$

Combinatorial proof for extended part

If $m+1$ element belongs to a j-cycle of σ_1, then

$$c_{j-1} wt(\sigma_1) = c_j wt(\sigma_1') \tag{2.10}$$

holds. We have two formulas for b_{m+1} and $(m+1)b_m$: let $(m)_k$ denote the falling factorial $m(m-1)\cdots(m-k+1)$, then

$$b_{m+1} = \sum_{j=1}^{m+1} (m)_{j-1} c_j b_{m+1-j}, \tag{2.11}$$
$$(m+1) b_m = \sum_{j=1}^{m+1} (m)_{j-1} c_{j-1} b_{m+1-j}, \tag{2.12}$$

both of which have combinatorial interpretation.

Combinatorial proof for extended part

Combinatorial interpretation.

For (2.11), jth term in the sum, $(m)_{j-1} c_j b_{m-j}$ implies the sum of $wt(\sigma_1)$ over all combinations in σ_1 such that j-cycle contains $m+1$th element. There are $(m)_{j-1}$ ways to construct j-cycle which contains $m+1$th element[1] and b_{m-j} is the sum of weights over all permutations for remaining $m-j$ elements. Here c_j is the weight of j-cycle.

For (2.12), jth term implies the sum of $wt(\sigma_1')$ over all permutations in σ_1 such that $m+1$th element is removed from j-cycle of σ_1, so that c_j of $(m)_{j-1} c_j b_{m-j}$ in (2.11) is replaced by c_{j-1}. Here we additionally use (2.9).

[1] There are ${}_mC_{j-1}$ ways to choose $j-1$ member of $j-$cycle and from this we construct $j!/j$ different cycles. Then ${}_mC_{j-1} \cdot j!/j = (m)_{j-1}$.

Combinatorial proof for extended part

Our goal is to prove (1.5) (⇔ (1.4)) by the induction. Since $b_0 = 1$, $b_1 = c_1$ and $b_2 = c_1^2 + c_2$, we have $b_0 b_2 - 2b_1^2 = c_2 - c_1^2 \geq 0$.
Assume that (1.5) holds with $k = m - 1$ and consider

$$\begin{aligned} & c_m(mb_{m-1}b_{m+1} - (m+1)b_m^2) \\ &= mb_{m-1}(c_m b_{m+1} - c_{m+1}(m+1)b_m) \\ &\quad - (m+1)b_m(c_m b_m - c_{m+1} m b_{m-1}). \end{aligned}$$

We use (2.11) and (2.12), i.e.

$$b_{m+1} = \sum_{j=1}^{m+1} (m)_{j-1} c_j b_{m+1-j},$$
$$(m+1)b_m = \sum_{j=1}^{m+1} (m)_{j-1} c_{j-1} b_{m+1-j}.$$

Combinatorial proof for extended part

$$\begin{aligned} & c_m(mb_{m-1}b_{m+1} - (m+1)b_m^2) \\ &= mb_{m-1}\Big(\sum_{j=1}^{m+1} c_m (m)_{j-1} c_j b_{m+1-j} - c_{m+1}(m)_{j-1} c_{j-1} b_{m+1-j}\Big) \\ &\quad - (m+1)b_m\Big(\sum_{j=1}^{m} c_m (m-1)_{j-1} c_j b_{m-j} - c_{m+1}(m-1)_{j-1} c_{j-1} b_{m-j}\Big) \\ &= mb_{m-1}\sum_{j=1}^{m} (m)_{j-1} b_{m+1-j}(c_m c_j - c_{m+1}c_{j-1}) \\ &\quad - mb_m \sum_{j=1}^{m} (m-1)_{j-1} b_{m-j}(c_m c_j - c_{m+1}c_{j-1}) \\ &\quad + b_m \sum_{j=1}^{m} (m-1)_{j-1} b_{m-j}(c_{m+1}c_{j-1} - c_m c_j) \\ &= m\sum_{j=1}^{m} (c_{m+1}c_{j-1} - c_m c_j)(m-1)_{j-2}\{(m+1-j)b_m b_{m-j} - mb_{m-1}b_{m+1-j}\} \\ &\quad + b_m \sum_{j=1}^{m} (m-1)_{j-1} b_{m-j}(c_{m+1}c_{j-1} - c_m c_j). \end{aligned}$$

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Extension 2 of BC (1996)

Let $c_0 = 1$ and let $c_1, c_2, \ldots$ be indeterminates. Further let

$$\begin{aligned} \mathscr{X} &= \{c_1, c_2, \ldots\}, \\ \mathscr{Y} &= \mathscr{X} \cup \{c_j c_k - c_{j-1}c_{k+1} : 0 < j \leq k\}, \\ \mathscr{Z} &= \mathscr{X} \cup \{c_{j-1}c_{k+1} - c_j c_k : 0 < j \leq k\} \end{aligned}$$

and define the sequence $(b_k)_{k\geq 0}$ by (1.1). Then

$$(n+1)b_m b_n - mb_{m-1}b_{n+1} \in \mathbb{N}[\mathscr{Y}], \quad (3.13)$$
$$mb_{m-1}b_{n+1} - (n+1)b_m b_n \in \mathbb{N}[\mathscr{Z}]/\mathbb{N}[\mathscr{X}] \quad \text{for} \quad 1 \leq m \leq n, \quad (3.14)$$

Remark: (3.13) is known and (3.14) is new.

Proof of extension 2 of BC (1996)

We show (3.14) by the induction. It is immediate to see

$$b_0 b_2 - 2b_1^2 = c_2 - c_1^2 \in \mathbb{N}[\mathscr{X}].$$

Assume $kb_{k-1}b_{k+1} - (k+1)b_k^2 \in \mathbb{N}[\mathscr{Z}]/\mathbb{N}[\mathscr{X}], \quad k \leq m-1$

then $kb_{k-1}b_{\ell+1} - (\ell+1)b_k b_\ell \in \mathbb{N}[\mathscr{Z}]/\mathbb{N}[\mathscr{X}], \quad k \leq \ell \leq m-1$

Indeed, we observe that

$$\frac{b_{\ell+1}}{(\ell+1)b_\ell} - \frac{b_k}{kb_{k-1}} = \Big(\frac{b_{\ell+1}}{(\ell+1)b_\ell} - \frac{b_\ell}{\ell b_{\ell-1}}\Big) + \cdots + \Big(\frac{b_{k+1}}{(k+1)b_k} - \frac{b_k}{kb_{k-1}}\Big) \in \frac{\mathbb{N}[\mathscr{Z}]}{\mathbb{N}[\mathscr{X}]}$$

Here $b_k \in \mathbb{N}[\mathscr{X}]$, $k \leq m-1$.

Proof of extension 2 of BC (1996)

Now we recall the previous equality in the proof of Theorem the first theorem :

$$\begin{aligned} & c_m(mb_{m-1}b_{m+1} - (m+1)b_m^2) \\ &= m\sum_{j=1}^{m} (m-1)_{j-2}(c_{m+1}c_{j-1} - c_m c_j) \\ &\quad \times \{(m+1-j)b_{m-j}b_m - mb_{m-1}b_{m+1-j}\}(*) \\ &\quad + b_m \sum_{j=1}^{m} (m-1)_{j-1} b_{m-j}(c_{m+1}c_{j-1} - c_m c_j). \end{aligned}$$

Then since $(*)$ holds for $1 \leq k \leq \ell \leq m-1$ by the assumption we conclude the induction hypothesis with $k = m$.

Normal and binomial convolutions (Schirmacher (1999))

For $i = 1, \dots, w$, $w \in \mathbb{N}$, $c_{i,0} = 1$ and $c_{i,1}, c_{i,2}, \dots$ indeterminates.

Let $\mathscr{Y}_i = \{c_{i,1}, c_{i,2}, \dots\} \cup \{c_{i,j}c_{i,k} - c_{i,j-1}c_{i,k+1} : 0 < j \le k\}$

and let
$$\sum_{k=0}^{\infty} A_k u^k = \sum_{k=0}^{\infty} \frac{B_k u^k}{k!} = \exp\Big(\sum_{j=0}^{\infty} \frac{\sum_{i=1}^{w} c_{i,j}}{j} u^j \Big). \quad (3.15)$$

Then for $1 \le m \le n$,
$$A_m A_n - A_{m-1}A_{n+1} \in \mathbb{R}_+[\cup_{i=1}^{w} \mathscr{Y}_i], \quad (3.16)$$
$$(n+1)A_{m-1}A_{n+1} - mA_mA_n \in \mathbb{R}_+[\cup_{i=1}^{w} \mathscr{Y}_i]. \quad (3.17)$$

The differences (3.16) and (3.17) are equivalent to
$$(n+1)B_mB_n - mB_{m-1}B_{n+1} \in \mathbb{R}_+[\cup_{i=1}^{w} \mathscr{Y}_i], \quad (3.18)$$
$$B_{m-1}B_{n+1} - B_mB_n \in \mathbb{R}_+[\cup_{i=1}^{w} \mathscr{Y}_i]. \quad (3.19)$$

Remarks

- There are several proofs:
  1. proof by direct calculations.
  2. proof with the Cauchy-Binet formula.
- Notice that the result of convolution theorem is not covered by the original version (log-concave) since the sum $C_j = \sum_{i=1}^{w} c_{i,j}$ in the relation
$$\sum_{k=0}^{\infty} A_k u^k = \sum_{k=0}^{\infty} \frac{B_k u^k}{k!} = \exp\Big(\sum_{j=0}^{\infty} \frac{C_j}{j} u^j \Big)$$
does not always log-concave. Namely, operations of (normal or binomial) convolution to sequences which satisfy (1.1) widens the class of sequence (c_j) from which log-concavity of (a_k) follows.

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Log-convex counterpart

For $w \in \mathbb{N}$ let
$$\sum_{k=0}^{\infty} A_k u^k = \sum_{k=0}^{\infty} \frac{B_k u^k}{k!} = \exp\Big(\sum_{j=1}^{\infty} \frac{\sum_{i=1}^{w} c_{i,j}}{j} u^j \Big).$$

Suppose that $(c_{i,k})_{k \ge 1}$, $i = 1, \dots, w$ are respectively log-convex in k. Then for $1 \le m \le n$
$$mB_{m-1}B_{n+1} - (n+1)B_mB_n \ge 0 \quad (4.20)$$
if and only if
$$\sum_{i=1}^{w} c_{i,2} - \Big(\sum_{i=1}^{w} c_{i,1} \Big)^2 \ge 0. \quad (4.21)$$

Equation (4.20) is equivalent to
$$A_{m-1}A_{n+1} - A_mA_n \ge 0.$$

Proof of log-convex counterpart

Let $C_0 = 1$ and $C_j = \sum_{i=1}^{w} c_{i,j}$. Put $\mathscr{V} = \{C_1, C_2, \dots\}$ and
$$\mathscr{W} = \mathscr{V} \cup \{C_{j-1}C_{k+1} - C_jC_k : 0 < j \le k\}.$$

Then the relation (1.1) concludes via the first theorem that
$$mB_{m-1}B_{n+1} - (n+1)B_mB_n \in \frac{\mathbb{N}[\mathscr{W}]}{\mathbb{N}[\mathscr{V}]} \quad \text{for} \quad 1 \le m \le n.$$

$(C_k)_{k \ge 1}$ is again log-convex (preserved under the summation).
$$C_0C_2 - C_1^2 = \sum_{i=1}^{w} c_{i,2} - \Big(\sum_{i=1}^{w} c_{i,1} \Big)^2 \ge 0$$
implies $C_0C_{k+1} - C_1C_k \ge 0,\ k \ge 1$, we conclude
$$mB_{m-1}B_{n+1} - (n+1)B_mB_n \ge 0 \quad \Leftrightarrow \quad A_{m-1}A_{n+1} - A_mA_n \ge 0.$$

Remarks

- The log-concavity is preserved by both ordinary and binomial convolution, while log-convexity is only by binomial convolution.
- Under the condition of the theorem (log-convex counter part) log-convexity is preserved by convolution.
- CI of the symm. group $Z(\Sigma_n)$ is relevant to ID dist. which satisfies
$$Z(\Sigma_n) = n^{-1} \sum_{k=1}^{n} c_k Z(\Sigma_{n-k}),$$
while P_k, $k \in \mathbb{Z}_+$ with $P_0 > 0$ is ID if and only if the quantity r_k with $k \in Z_+$ determined by
$$P_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} r_k P_{n-k} \quad (4.22)$$
are non-negative.

What is interpretation for the CI of sym. group and ID ?

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Thank you for your attention !