The Simultaneous Multivariate Hawkes-type Point Processes and their application to Financial Markets

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1. In economic and financial time series we sometimes observe large jumps. Although they are relatively rare events, they have significant influence not only on a financial market but also several different markets and macro economies. By using the simultaneous Hawkes models we introduce, which are multivariate point processes, it is possible to analyze the causal effects of large events in the sense of the Granger-non-causality (GNC) and the instantaneous Granger-non-causality (IGNC). We investigate the financial market of Tokyo and other markets, and apply the Granger non-causality tests. We have found several important empirical findings among financial markets and macro economies.

2. We divide the continuous observation period $[0,T]$ to the discrete observation periods $I_i^n = (t_{i-1}^n, t_i^n]$ $(i=1,\cdots,n)$. The initial time is $t_0^n = 0$ and we interpret I_i^n as the $i-$th day, but it may be possible to more finer observations. Let the observable $d-$dimension stochastic process be $P_j(t)$ $(j=1,\cdots,d\ ; t_{i-1}^n < t \leq t_i^n, i=1,\cdots,n)$ and in $s \in I_i$ we consider the (negative) log-returns of prices $Y_j^n(s)$ $(t_{i-1}^n < s \leq t_i^n)$ be

$$(1) \qquad Y_j^n(s) = -\log[P_j(s)/P_j(t_{i-1}^n)] \ (j=1,\cdots,d; i=1,\cdots,n) \ .$$

Let the first stopping time when $Y_j^n(s)$ exceeds the threshold u_j in $s \in I_i$ be $\tau_j^n(i,1)$. We define $X_j^n(s) = Y_j^n(s)$ for $s \in t_{i-1}^n \leq s \leq \tau_j^n(i,1)$ and $X_j^n(s) = X_j^n(\tau_j^n(i,1))$ for $s \in [\tau_j^n(i,1), t_i^n]$.

We define the simple counting processes $N_j^{n*}(s,u_k)$ by the number of stopping times that $X_j^n(s)$ exceed u_j $(j=1,\cdots,d)$ for a particular j but not for other $k \neq j$ by the time s. For the simplicity, we assume that the jumps of the counting process $N_j^{n*}(s,u_k)$ can occur at t_i^n the end of each intervals $(t_{i-1}^n, t_i^n]$. Then we notice that as $n \longrightarrow \infty$ the interval length goes to zero, that is, $\max_{i=1,\cdots,n} |t_i^n - t_{i-1}^n| \longrightarrow 0$ and the simple counting process $N_j^{n*}(s,u_k)$ converges to $N_j^{*}(s,u_k)$. The resulting counting process can be interpreted as the limiting process in the high frequency asymptotics. (Ait-Sahalia and Jacod (2014), for instance.)

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The simple point process we consider $N_j^*(t)$ $(j = 1, \cdots, d)$ satisfies the standard conditions that as $\Delta t \to 0$

$$
\begin{gathered}
P(N_j^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u) = 1|\mathcal{F}_t^n) = \lambda_j^n(t,u_j)\Delta t + o_p(\Delta t) \ , \\
P(N_j^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u) > 1|\mathcal{F}_t^n) = o_p(\Delta t) \\
P(N_k^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u_j) \geq 1|\mathcal{F}_t^n) = o_p(\Delta t) \text{ for } k \neq j \ ,
\end{gathered}
$$

where $\mathcal{F}_t^n$ is the $\sigma-$field generated by the information at t and the (conditional) intensity functions are given by

$$
(2) \qquad \lambda_j^{n*}(t,u_j) = \lim_{\Delta t \to 0} \mathcal{E}[\frac{N_j^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u_j)}{\Delta t}|\mathcal{F}_{t-}^n] \ .
$$

(We denote $\mathcal{F}_t$ for $\mathcal{F}_{t-}^n$ whenever there is no confusion.)

Also we define the simple proint processes $N_{jk}^{n*}(s,u_{jk})$ by the number of stopping times that $X_j^n(s)$ exceed u_j $(j = 1, \cdots, d)$ for a particular j and $X_k^n(s)$ exceed u_k $(k = 1, \cdots, d; k \neq j)$ for a particular k and other $X_l^n(s)$ $(l \neq j,k)$ do not exceed u_l by the time s in I_i^n. Then we introduce the point processes $N_{jk}^{n*}(t,u_{jk})$ with co-jumps of N_j and N_k by

$$
\begin{gathered}
P(N_j^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u_j) = N_k^{n*}(t+\Delta t, u_k) - N_k^{n*}(t,u_k) = 1|\mathcal{F}_t) \\
= \lambda_{jk}^{n*}(t,u_{jk})\Delta t + o_p(\Delta t) \ , \\
P(N_j^{n*}(t+\Delta t, u_j) - N_j^{n*}(t,u_j) > 1|\mathcal{F}_t) = o_p(\Delta t) \\
P(N_k^{n*}(t+\Delta t, u_k) - N_j^{n*}(t,u_k) > |\mathcal{F}_t) = o_p(\Delta t) \text{ for } k \neq j \ .
\end{gathered}
$$

When we have co-jumps of two point processes, we can define the point process

$$
(3) \qquad N_j^n(s,u_j) = N_j^{n*}(s,u_j) + N_{j,k}^{n*}(s,u_{jk}) \ (j \neq k; j,k = 1, \cdots, d)
$$

and the corresponding conditional intensity functions is given by

$$
(4) \qquad \lambda_j^n(t,u_j) = \lambda_j^{n*}(t,u_j) + \lambda_{j,k}^{n*}(t,u_{jk}) \ .
$$

Then the resulting can be interpreted as the marginal point process of the j-th component of the vector point process $\mathbf{N}^n(s,\mathbf{u})$. It is straightforward to extend this formulation to more complicated co-jumps. There are one-to-one transformations between $N_j^n(s,u_j)$ and $N_j^{n*}(s,u_j)$, and $\lambda_j^n(t,u_j)$ and $\lambda_j^{n*}(t,u_j)$. It is straight-forward to extend this construction to the simultaneous jumps more than two components in a straightforward way. In the following notation we can choose the same $u_j = u$ $(j = 1, \cdots, d)$, but they can be different in principle.

We shall consider the self-exciting form of intensity function as

$$
(5) \qquad \lambda_j^{n*}(t,u) = \lambda_{j0}^* + \int_{-\infty}^t \sum_{i=1}^p c_{ji}(X^n(s-))g_{ji}(t-s)dN_{J_i}^{n*}(s,u) \ ,
$$

where the index set is defined as $J_i = i$ $(i = 1, \cdots, d)$, $J_i = 1, 1 + (i-d)$ $(i = d+1, \cdots, 2d-1), \cdots$, $J_p = 1, \cdots, d$. That is, we sequentially define $N_i^{n*}(s,u_i) =$

$N_i^{n*}(s,u)$ $(i=1,\cdots,d)$, $N_{d+1}^{n*}(s,u)=N_{1,2}^{n*}(s,u),\cdots$, and $N_p^{n*}(s,u)=N_{1,\cdots,d}^{n*}(s,u)$. In this formulation we will use the discounting function as $g_{ji}(t-s)=e^{-\gamma_{ji}(t-s)}$ and the impact function as $C(X)=(c_{ji}(x))$.

Since we are interested in large jumps, it is important to use the probability function in the tails. We adopt the tail probability function for $x>u_j$ $(j=1,\cdots,d)$ as

$$
(6) \qquad P(X_j^n(s)>x|X_j^n(s)>u_j,\mathcal{F}_s) = \frac{\left[1+\frac{\xi_j}{\sigma_j(s)}x\right]^{-1/\xi_j}}{\left[1+\frac{\xi_j}{\sigma_j(s)}u_j\right]^{-1/\xi_j}} = \left[1+\frac{\xi_j}{\sigma_j^*(s)}(x-u_j)\right]^{-1/\xi_j},
$$

and we set $\sigma_j^*(s)=\xi_j u_j+\sigma_j(s)$. We assume that given the return at s $X_j^n(s)$ the conditional density function is given by

$$
(7) \qquad f_j(x,s)=\frac{1}{\sigma_j^*(s)}\left[1+\frac{\xi_j}{\sigma_j^*(s)}(x-u_j)\right]^{-1/\xi_j-1} \quad (x>u_j)\ .
$$

(See Resnick (2007) for the generalized Pareto distribution as the statistical extreme value theory.
As the general formulation, it is possible to use the conditional intensity function as

$$
\begin{aligned}
(8) \qquad \lambda_j^n(t,u) &= \lambda_{j0}^n+\sum_{i=1}^{d}\int_0^t[A_{ji}(X_i^n)^c(s-)]g_i(t-s)dN_i^n(s,u) \\
&\quad +\sum_{i=d+1}^{p}\int_0^t[A_{ji}\max_{k\in J_i}(X_k^n)^c(s-)]g_i(t-s)dN_{J_i}^n(s,u)\ ,
\end{aligned}
$$

where $N_{d+1}^n(s,u)=N_{12}^n(s,u),\cdots$, $N_p^n(s,u)=N_{1\cdots d}^n(s,u)$ and the parameters λ_{j0}, A_{ji}, and γ_i are constants. We denote the impact fucntion $C_{ji}(x_j)=A_{ji}x_j^c$ $(i=1,\cdots,d)$. In particular when $p=d$ and $C_{ji}=\delta(j,i)$(indicator function), it corresponds to the multivariate Hawkes process, which is a simple point process without co-jumps.

Let $p\times p$ transformation matrix as

$$
\mathbf{D}=\begin{bmatrix}\mathbf{D}_1\\ \mathbf{D}_2\end{bmatrix}=\begin{bmatrix}
1 & 0 & \cdots & & 0 & 1 & 1 & \cdots & 0 & \cdots & 1\\
0 & 1 & \cdots & & 0 & 1 & 0 & \cdots & 0 & \cdots & 1\\
\vdots & & & & 1 & 0 & & & \vdots & \cdots & 1\\
\vdots & & & & 0 & 1 & & & \vdots & \cdots & 0\\
0 & \cdots & & 0 & 0 & & \cdots & & 0 & \cdots & 1
\end{bmatrix},
$$

where $\mathbf{D}_1$ is a $d \times p$ matrix, $\mathbf{D}_2$ is a $(p-d) \times p$ matrix and $p = 2^d - 1$. Also let $p \times 1$ vectors

$$\lambda^n(t,\mathbf{u}) = \begin{bmatrix} \lambda_1^n(t,u_1) \\ \vdots \\ \lambda_d^n(t,u_p) \\ \lambda_{12}^n(t,u_1) \\ \vdots \\ \lambda_{12\cdots d}^n(t,u_p) \end{bmatrix} , \ \mathbf{N}(t,\mathbf{u}) = \begin{bmatrix} N_1(t,u_1) \\ \vdots \\ N_p(t,u_p) \\ N_{12}(t,u_1) \\ \vdots \\ N_{12\cdots d}(t,u_p) \end{bmatrix} ,$$

and $p \times p$ matrices

$$\mathbf{C}(X(s-)) = [c_{ij}(X_{s-})] \ , \ \mathbf{G}(t-s) = [diag(g_j(t-s))] \ .$$

We call the resulting Hawkes-type intensity models as the simultaneous Hawkes-type point process (SHPP) models because the resulting point processes are not necessarily simple [5] . The classical Hawkes point processes have been useful because they are simple point processes, but they exclude the possibility of simultaneous jumps or co-jumps. Our constructions are some extensions of Solo (2007).

3. **References**

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