Estimation of the shape of density level sets of star-shaped distributions

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Abstract

Elliptically contoured distributions generalize the multivariate normal distributions in such a way that the density generators need not be exponential. However, as the name suggests, elliptically contoured distributions remain to be restricted in that the similar density contours need to be elliptical. Kamiya, Takemura and Kuriki [Star-shaped distributions and their generalizations, *Journal of Statistical Planning and Inference* **138** (2008), 3429–3447] proposed star-shaped distributions in which the density level sets are allowed to be arbitrary similar star-shaped sets. In this paper, we propose a nonparametric estimator of the shape of the density contours of star-shaped distributions, and prove its strong consistency with respect to the Hausdorff distance. We examine the performance of our estimator by simulation studies.

Key words: density contour, direction, elliptically contoured distribution, Hausdorff distance, kernel density estimator, star-shaped distribution, strong consistency.

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1 Introduction

Elliptically contoured distributions generalize the multivariate normal distributions in such a way that the density generators need not be exponential. In this way, the class of elliptically contoured distributions includes, for example, distributions whose tails are heavier than those of the multivariate normal distributions. However, as the name suggests, elliptically contoured distributions remain to be restricted in that the similar density contours need to be elliptical. Hence, skewed distributions are not included in this class.

Kamiya, Takemura and Kuriki [5] proposed star-shaped distributions in which the density level sets are allowed to be arbitrary similar star-shaped sets (see also [6], [4]). Essentially the same idea can be found in v-spherical distributions by Fernández, Osiewalski and Steel [2] and center-similar distributions by Yang and Kotz [7]. Asymmetry is allowed in star-shaped distributions. Hence, besides distributions which are symmetric with respect to the center such as elliptically contoured distributions and l_q-spherical distributions, the class of star-shaped distributions includes asymmetric distributions such as multivariate skewed exponential power distributions.

Kamiya, Takemura and Kuriki [5] studied distributional properties of star-shaped distributions, e.g., the independence of the "length" and the "direction," and the robustness of the distribution of the "direction." However, they did not investigate inferential problems about star-shaped distributions. From the perspective of [5], the most important problem in the inference for star-shaped distributions is the estimation of the shape of their density contours

In this paper, we propose a nonparametric estimator of the shape of the density contours. The point is that the density of the usual direction under a star-shaped distribution is in one-to-one correspondence with a function which determines the shape of the density contours. Thus, by nonparametrically estimating the density of the direction, we can obtain a nonparametric estimator of the shape. We prove its strong consistency with respect to the Hausdorff distance.

The organization of this paper is as follows. We describe a star-shaped distribution and define the shape of its density contours in Section 2. Next, in Section 3 we propose an estimator of the shape of the density contours of a star-shaped distribution. Next, in Section 4 we prove strong consistency of our estimator of the shape. In Section 5, we examine the performance of our estimator by simulation studies. We conclude with some remarks in Section 6.

2 Star-shaped distribution and the shape of its density contours

In this section, we describe a star-shaped distribution and define the shape of its density contours.

Suppose a random vector $x \in \mathcal{X} := \mathbb{R}^p \setminus \{\mathbf{0}\},\ p \geq 2$, is distributed as

$$x \sim h(r(x))dx, \tag{1}$$

where $r : \mathcal{X} \to \mathbb{R}_{>0}$ is continuous and equivariant under the action of the positive real numbers: $r(cx) = cr(x)$ for all $c \in \mathbb{R}_{>0}$. In the particular case that $r(x) = (x^T \Sigma^{-1} x)^{1/2}$ ($\cdot^T$ denotes the transpose) for a positive definite matrix Σ and that the density generator $h((-2(\,\cdot\,))^{1/2})$ is exponential: $h((-2(\,\cdot\,))^{1/2}) = \text{const} \times \exp(\,\cdot\,)$, we obtain the multivariate normal distribution $N_p(\mathbf{0}, \Sigma)$.

Define $Z := \{x \in \mathcal{X} : r(x) = 1\}$, and write $cZ := \{cz : z \in Z\}$ for $c \in \mathbb{R}_{\geq 0}$. Then the density $h(r(x))$ is constant on each of $cZ \subset \mathcal{X},\ c \in \mathbb{R}_{>0}$: $h(r(x)) = h(c)$ for all $x \in cZ$. In cases where $h : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is injective (e.g., strictly decreasing), each $cZ,\ c \in \mathbb{R}_{>0}$, is a contour of the density $h(r(x))$: $cZ = \{x \in \mathcal{X} : h(r(x)) = h(c)\}$, but in general, a contour of the density is a union of some cZ's: $\{x \in \mathcal{X} : h(r(x)) = t\} = \bigcup_{c \in h^{-1}(\{t\})} cZ,\ t \in \mathbb{R}_{\geq 0}$.

Noticing that $\mathcal{Z} := \bigcup_{0 \leq c \leq 1} cZ \subset \mathcal{X} \cup \{\mathbf{0}\} = \mathbb{R}^p$ is a star-shaped set with respect to the origin, we say that x in (1) has a star-shaped distribution. Also, we call Z the shape of the density contours of this star-shaped distribution, including cases where h is not injective.

3 Estimation of the shape

In this section, we propose an estimator of the shape Z of the density contours of a star-shaped distribution.

Let $\|\cdot\|$ denote the Euclidean norm. Under (1), the direction $u := x/\|x\| \in \mathbb{S}^{p-1}$ (the unit sphere in $\mathbb{R}^p$) is distributed as

$$u \sim f(u)du \quad \text{with} \quad f(u) := c_0 r(u)^{-p}, \tag{2}$$

where du stands for the volume element of $\mathbb{S}^{p-1}$ and $c_0 = 1/\int_{\mathbb{S}^{p-1}} r(u)^{-p} du$ (Theorem 4.1 of [5]). Note the function $f : \mathbb{S}^{p-1} \to \mathbb{R}_{\geq 0}$ in (2) is continuous and satisfies $f(u) > 0$ for all $u \in \mathbb{S}^{p-1}$. From now on, we assume r is taken so that $\int_{\mathbb{S}^{p-1}} r(u)^{-p} du = 1$ and hence $c_0 = 1$.

Now, we can write $r(u) = f(u)^{-1/p}$ for $u \in \mathbb{S}^{p-1}$. Thus, for $x \in \mathcal{X}$, the condition that $r(x) = 1$ is equivalent to $\|x\| = 1/r(x/\|x\|) = f(x/\|x\|)^{1/p}$. Hence

$$Z = \{x \in \mathcal{X} : r(x) = 1\} = \left\{ f(u)^{\frac{1}{p}} u : u \in \mathbb{S}^{p-1} \right\},$$

and we can estimate Z by estimating the density $f(u)$ of $u = x/\|x\|$.

Suppose we are given an i.i.d. sample $x_1, \ldots, x_n$ from (1), and consider estimating $f(u)$ based on $u_1, \ldots, u_n$, where $u_i := x_i/\|x_i\|,\ i = 1, \ldots, n$.

Let $\hat{f}_n(u)$ be an estimator of $f(u)$ such that $\hat{f}_n(u) \geq 0$ for all $u \in \mathbb{S}^{p-1}$. Define the estimator $\hat{Z}_n$ of Z by

$$\hat{Z}_n := \left\{ \hat{f}_n(u)^{\frac{1}{p}} u : u \in \mathbb{S}^{p-1} \right\}.$$

Then $\hat{\mathcal{Z}}_n := \bigcup_{0 \leq c \leq 1} c\hat{Z}_n$ is also a star-shaped set with respect to the origin.

4 Strong consistency

In this section, we prove strong consistency of our estimator $\hat{Z}_n$ of the shape Z.

Let $\delta_H(\hat{Z}_n, Z)$ be the Hausdorff distance between $\hat{Z}_n$ and Z:

$$\delta_H(\hat{Z}_n, Z) := \inf \left\{ \delta > 0 : \hat{Z}_n \subset Z + B(\delta),\ Z \subset \hat{Z}_n + B(\delta) \right\},$$

where $B(\delta) := \{x \in \mathbb{R}^p : \|x\| \leq \delta\}$, and $+$ denotes the Minkowski sum. Similarly, let

$$\delta_H(\hat{\mathcal{Z}}_n, \mathcal{Z}) = \inf \left\{ \delta > 0 : \hat{\mathcal{Z}}_n \subset \mathcal{Z} + B(\delta),\ \mathcal{Z} \subset \hat{\mathcal{Z}}_n + B(\delta) \right\}.$$

The purpose of this section is to show that, under some conditions, $\delta_H(\hat{Z}_n, Z)$ and $\delta_H(\hat{\mathcal{Z}}_n, \mathcal{Z})$ converge to zero almost surely.

We begin by proving that $\delta_H(\hat{Z}_n, Z)$ and $\delta_H(\hat{\mathcal{Z}}_n, \mathcal{Z})$ are bounded by $d_n := \sup_{u \in \mathbb{S}^{p-1}} |\hat{f}_n(u)^{1/p} - f(u)^{1/p}|$:

$$\delta_H(\hat{Z}_n, Z) \leq d_n, \qquad \delta_H(\hat{\mathcal{Z}}_n, \mathcal{Z}) \leq d_n. \tag{3}$$

Let $z_0 = \tilde{c}_0 f(u_0)^{1/p} u_0$ $(0 \leq \tilde{c}_0 \leq 1,\ u_0 \in \mathbb{S}^{p-1})$ be an arbitrary point of $\mathcal{Z}$. Take $z_0' = \tilde{c}_0 \hat{f}_n(u_0)^{1/p} u_0 \in \hat{\mathcal{Z}}_n$. Then $\|z_0' - z_0\| = \tilde{c}_0 |\hat{f}_n(u_0)^{1/p} - f(u_0)^{1/p}| \leq d_n$, and thus $z_0 \in \hat{\mathcal{Z}}_n + B(d_n)$. This argument implies that $\mathcal{Z} \subset \hat{\mathcal{Z}}_n + B(d_n)$. Similarly, $\hat{\mathcal{Z}}_n \subset \mathcal{Z} + B(d_n)$ holds true. Therefore, the second inequality in (3) is proved. The proof of the first inequality in (3) is similar.

Next we want to verify that $d_n \to 0$ $(n \to \infty)$ almost surely for estimators $\hat{f}_n(u)$ having a certain property.

For each $u \in \mathbb{S}^{p-1}$ and each n, we can write

$$\hat{f}_n(u)^{\frac{1}{p}} = f(u)^{\frac{1}{p}} + \frac{1}{p} f_n^*(u)^{\frac{1}{p}-1} \Big(\hat{f}_n(u) - f(u) \Big) \tag{4}$$

for some $f_n^*(u)$ between $\hat{f}_n(u)$ and $f(u)$.

Let $\epsilon_n := \sup_{u \in \mathbb{S}^{p-1}} |\hat{f}_n(u) - f(u)|$. Then we have $f_n^*(u) \geq f(u) - \epsilon_n$ for all $u \in \mathbb{S}^{p-1}$ and all n, and thus

$$\inf_{u \in \mathbb{S}^{p-1}} f_n^*(u) \geq \inf_{u \in \mathbb{S}^{p-1}} f(u) - \epsilon_n \tag{5}$$

for all n. Since $f : \mathbb{S}^{p-1} \to \mathbb{R}_{\geq 0}$ is continuous, $\mathbb{S}^{p-1}$ is compact and $f(u) > 0$ for all $u \in \mathbb{S}^{p-1}$, we have $c_f := \inf_{u \in \mathbb{S}^{p-1}} f(u) = \min_{u \in \mathbb{S}^{p-1}} f(u) > 0$. Now, suppose the estimator $\hat{f}_n(u)$ satisfies

$$\epsilon_n = \sup_{u \in \mathbb{S}^{p-1}} \Big| \hat{f}_n(u) - f(u) \Big| \to 0 \quad \text{a.s.} \tag{6}$$

Then, with probability one, we have $\epsilon_n < c_f/2$ for all sufficiently large n. Together with this fact, inequality (5) implies that, with probability one,

$$\inf_{u \in \mathbb{S}^{p-1}} f_n^*(u) \geq c_f - \epsilon_n > \frac{c_f}{2} \tag{7}$$

for all sufficiently large n.

It follows from (4) and (7) that, with probability one,

$$d_n = \sup_{u\in\mathbb{S}^{p-1}} \left|\hat{f}_n(u)^{\frac{1}{p}} - f(u)^{\frac{1}{p}}\right| \le \frac{1}{p}\left\{\inf_{u\in\mathbb{S}^{p-1}} f_n^*(u)\right\}^{\frac{1}{p}-1} \cdot \sup_{u\in\mathbb{S}^{p-1}} \left|\hat{f}_n(u) - f(u)\right|$$
$$\le \frac{1}{p}\left(\frac{c_f}{2}\right)^{\frac{1}{p}-1}\epsilon_n$$

for all sufficiently large n. Therefore, by (6) we obtain $d_n \to 0$ $(n\to\infty)$ almost surely, as was to be verified.

Now, for estimating a general density $f(u)$ on $\mathbb{S}^{p-1}$, $p \ge 2$ (i.e., not necessarily $f(u)$ in (2)) based on an i.i.d. sample $u_1,\dots,u_n$ from $f(u)du$, we can use the following kernel density estimator (Hall, Watson and Cabrera [3], Bai, Rao and Zhao [1]):

$$\hat{f}_n(u) = \frac{C(\eta)}{n\eta^{p-1}} \sum_{i=1}^{n} L\left(\frac{1-u^T u_i}{\eta^2}\right), \quad u \in \mathbb{S}^{p-1}, \tag{8}$$

where $\eta = \eta_n > 0$, $L : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ satisfies $0 < \int_0^\infty L(v)v^{(p-3)/2}dv < \infty$ and $C(\eta) > 0$ is given by $C(\eta) = \eta^{p-1}/\int_{\mathbb{S}^{p-1}} L((1-u^T y)/\eta^2)du$, $y \in \mathbb{S}^{p-1}$. Note that $C(\eta)$ does not depend on y and can be written as $C(\eta) = \eta^{p-1}/\{\omega_{p-1}\int_{-1}^{1} L((1-t)/\eta^2)(1-t^2)^{(p-3)/2}dt\} = 1/\{\omega_{p-1}\int_0^{2/\eta^2} L(v)v^{(p-3)/2}(2-v\eta^2)^{(p-3)/2}dv\}$, $\omega_{p-1} := 2\pi^{(p-1)/2}/\Gamma((p-1)/2)$ (equation (2·2) of [3]; equation (1.6) of [1]).

A sufficient condition for $\sup_{u\in\mathbb{S}^{p-1}} |\hat{f}_n(u) - f(u)| \to 0$ a.s. for a general density $f(u)$ on $\mathbb{S}^{p-1}$, $p \ge 2$, and its kernel estimator $\hat{f}_n(u)$ in (8) was obtained by Bai, Rao and Zhao [1], Theorem 2: $\sup_{u\in\mathbb{S}^{p-1}} |\hat{f}_n(u) - f(u)| \to 0$ a.s. holds true if the following conditions are satisfied: 1. $f : \mathbb{S}^{p-1} \to \mathbb{R}_{\ge 0}$ is continuous; 2. $L : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is bounded; 3. $L : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is Riemann integrable on any finite interval in $\mathbb{R}_{\ge 0}$ with $\int_0^\infty \sup_{w:\,|\sqrt{w}-\sqrt{v}|<1} L(w) \cdot v^{(p-3)/2}dv < \infty$; 4. $\eta_n \to 0$ as $n \to \infty$; 5. $n\eta_n^{p-1}/\log n \to \infty$ as $n \to \infty$.

Note that under the fourth condition $\eta_n \to 0$ $(n \to \infty)$, we have $\lim_{n\to\infty} C(\eta_n) = 1/\{2^{(p-3)/2}\omega_{p-1}\int_0^\infty L(v)v^{(p-3)/2}dv\}$ (equation (1.7) of [1]).

The preceding arguments yield the following result:

Theorem 4.1. *Let* $x_1, \ldots, x_n \in \mathcal{X} = \mathbb{R}^p \setminus \{\mathbf{0}\},\ p \geq 2,$ *be an i.i.d. sample from a star-shaped distribution* $h(r(x))dx$*. Let* $\hat{f}_n(u) = (C(\eta)/(n\eta^{p-1})) \sum_{i=1}^n L((1 - u^T u_i)/\eta^2)$ *be a kernel estimator of the density* $f(u)$ *of* $u = x/\|x\| \in \mathbb{S}^{p-1},\ x \sim h(r(x))dx,$ *based on* $u_i = x_i/\|x_i\|,\ i = 1, \ldots, n.$

Assume the equivariant function $r : \mathcal{X} \to \mathbb{R}_{>0}$ *under the action of the positive real numbers is continuous and normalized so that* $\int_{\mathbb{S}^{p-1}} r(u)^{-p} du = 1$*, and that* $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is bounded and satisfies* $\int_0^\infty L(v) v^{(p-3)/2} dv > 0$ *and* $\int_0^\infty \sup_{w:\, |\sqrt{w} - \sqrt{v}| < 1} L(w) \cdot v^{(p-3)/2} dv < \infty$*. Moreover, suppose* $\eta = \eta_n > 0$ *is taken in such a way that* $\eta_n \to 0$ *and* $n\eta_n^{p-1}/\log n \to \infty$ *as* $n \to \infty$*.*

Then, $\hat{Z}_n = \{\hat{f}_n(u)^{1/p} u : u \in \mathbb{S}^{p-1}\}$ *is a strongly consistent estimator of the shape* $Z = \{x \in \mathcal{X} : r(x) = 1\}$ *of the density contours of the star-shaped distribution in the sense that the Hausdorff distance* $\delta_H(\hat{Z}_n, Z)$ *between* $\hat{Z}_n$ *and* Z *satisfies*

$$\delta_H(\hat{Z}_n, Z) \to 0 \quad (n \to \infty) \quad \text{a.s.}$$

In addition, $\hat{\mathcal{Z}}_n = \bigcup_{0 \leq c \leq 1} c\hat{Z}_n$ *is a strongly consistent estimator of* $\mathcal{Z} = \bigcup_{0 \leq c \leq 1} cZ$*:* $\delta_H(\hat{\mathcal{Z}}_n, \mathcal{Z}) \to 0$ a.s.

It can easily be seen that $L(v) = e^{-v}$ and $L(v) = 1(v < 1)$ $(= 1$ if $v < 1$ and 0 otherwise) satisfy the conditions of Theorem 4.1.

5 Simulation studies

Results of simulation studies will be demonstrated in the talk.

6 Concluding remarks

In this paper, we proposed a nonparametric estimator of the shape of the density contours of star-shaped distributions, and proved its strong consistency with respect to the Hausdorff distance.

We can introduce the location parameter and consider a star-shaped distribution whose density level sets are star-shaped with respect to the location. In that case, one possibility for estimating the shape is to plug in an estimator of the location and use our proposed nonparametric estimator of the shape. We might be able to estimate the location by characterizing it in some way. For example, if the star-shaped distribution may be assumed to have symmetry with respect to the location and a finite first moment, the location can be characterized as the mean and may be estimated by, e.g., the sample mean. If, instead, $h(\,\cdot\,)$ in (1) is strictly decreasing, the location can be regarded as the mode and be estimated by means of various methods for estimating the multivariate mode.

References

[1] Z. D. Bai, C. R. Rao and L. C. Zhao, Kernel estimators of density function of directional data, *Journal of Multivariate Analysis* **27** (1988), 24–39.

[2] C. Fernández, J. Osiewalski and M. F. J. Steel, Modeling and inference with v-spherical distributions, *Journal of the American Statistical Association* **90** (1995), 1331–1340.

[3] P. Hall, G. S. Watson and J. Cabrera, Kernel density estimation with spherical data, *Biometrika* **74** (1987), 751–762.

[4] H. Kamiya and A. Takemura, Hierarchical orbital decompositions and extended decomposable distributions, *Journal of Multivariate Analysis* **99** (2008), 339–357.

[5] H. Kamiya, A. Takemura and S. Kuriki, Star-shaped distributions and their generalizations, *Journal of Statistical Planning and Inference* **138** (2008), 3429–3447.

[6] A. Takemura and S. Kuriki, Theory of cross sectionally contoured distributions and its applications, Discussion Paper Series 96-F-15, July 1996, Faculty of Economics, University of Tokyo.

[7] Z. Yang and S. Kotz, Center-similar distributions with applications in multivariate analysis, *Statistics & Probability Letters* **64** (2003), 335–345.