Optimal investment in correlated stocks and an index bond for defined contribution pension

Ryou YAMAGATA

Graduate Program of Science and Technology, Mathematics Division
Faculty of Science and Technology, Sophia University

1 Introduction

Recently in Japan, it is promoted that "from savings to investment" in individual asset management. 401K pension plan and Nippon individual savings account (NISA) go with the stream. There are two kinds of pension plans, one is defined contribution pension plan (DCP) and the other is defined benefit pension (DBP). We focus our attention to the former, and it is a type of pension plan which both employee and employer make contribution pay with fixed percentage of salary. The percentage is determined depending on the state of individual pension plan, and pension members receive the size of accumulation investment earnings at the retirement age. The difficulty is that pension members cannot know how much the retirement benefit is. Consequently DCP requires more precise and careful asset management than DBP.

Then the purpose of this research work is to derive the optimal investment plan to hedge risks and maximize the total wealth. The target of investments are domestic and foreign stocks, domestic and foreign bonds, Real Estate Investment Trust (REIT), insurance, trust, fund, and other financial products. We consider the problem where pension members invest in financial products including stocks and index bonds. On the assumption that pension members are not day-trader and their portfolios are rebalanced once a week at most, we derive a theoretical result of the optimal investment plan with numerical examples.

2 Theory of the optimal investment problem

2.1 Basic definition for optimal investment problem

Throughout the following discussion, we assume that the market is arbitrage-free and complete. Let $t \in [0, T]$ be continuous time variable provided that $t = 0$ is the starting time of DC plan, and $t = T$ is the terminal time of the pension. Then we define the stochastic price level as

$$\begin{aligned} \frac{dP(t)}{P(t)} &= idt + \sigma_{00} dW_0(t) \\ P(0) &= p > 0 \end{aligned}$$

where the constant i is the expected rate of inflation and $W_0(t)$ is a Brownian Motion. The volatility of the price level is σ_{00}. Suppose that the market consists of a risk-free bond (in the market money account), an index bond (or financial instrument like fund), and stocks (or other financial instruments), and they have the following properties:.

1) The price of risk-free an bond(in the market money account) with (constant) risk-free rate R is

$$\frac{dB(t)}{B(t)} = Rdt$$

2) The price process of index bond(or fund) with real return r satisfies

$$\begin{aligned} \frac{dI(t)}{I(t)} &= rdt + \frac{dP(t)}{P(t)} \\ &= (r + i)\, dt + \sigma_{00} dW_0(t) \end{aligned}$$

3) The price process of n stocks (or other financial instruments) are given by

$$\frac{dS(t)}{S(t)} = \mu dt + \Sigma\, dW_T(t)$$

In those notations,

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} \quad , \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}$$

$$S(t) = \begin{pmatrix} S_1(t) & 0 & \dots & 0 \\ 0 & S_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_n(t) \end{pmatrix} \quad , \quad dS(t) = \begin{pmatrix} dS_1(t) \\ dS_2(t) \\ \vdots \\ dS_n(t) \end{pmatrix}$$

$$W_T(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{pmatrix} \quad , \quad dW_T(t) = \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ \vdots \\ dW_n(t) \end{pmatrix}$$

where μ is the expected of return vector on the stocks (or other financial instruments), Σ is volatility matrix of stocks (or other financial instruments), and $W_T(t)$ is Brownian motion vector with $W_k \perp\!\!\!\perp W_j$ $(k \neq j,\ k, j = 0, 1, 2, \dots, n)$.

Let us assume that $\sigma_{kk} \neq 0$ and set the volatility matrix

$$\sigma \quad = \quad \begin{pmatrix} \sigma_{00} & 0 \\ 0 & \Sigma \end{pmatrix} = \begin{pmatrix} \sigma_{00} & 0 & 0 & \dots & 0 \\ 0 & \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ 0 & \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}$$

Then market price of risk θ is

$$\theta = \sigma^{-1} \begin{pmatrix} r + i - R \\ \mu - R \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta n \end{pmatrix}$$

Moreover, define the salary process of the pension plan member by

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= \gamma dt + \sigma_y dW_0(t) + \frac{dP(t)}{P(t)} \\ &= (\gamma + i)dt + (\sigma_y + \sigma_{00})dW_0(t) \qquad (1) \\ &= \kappa dt + \sigma_s dW(t) \\ \kappa = \gamma + i, &\qquad \sigma_s = \sigma_y + \sigma_{00} \\ Y(0) &= y > 0 \end{aligned}$$

where γ is the expected growth rate of salary, σ_y is the volatility of salary, and both are constant. As is observed in (1), the salary is linked with the expected rate of inflation in the long term. The salary process $Y(t)$ can be solved by using Ito's Lemma as follows.

$$Y(t) = y \exp\left\{ \left(\kappa - \frac{1}{2} \|\sigma_Y\|^2 \right) t + \sigma_Y^\top W(t) \right\}$$

where

$$\sigma_Y = \begin{pmatrix} \sigma_s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad W(t) = \begin{pmatrix} W_0(t) \\ W_T(t) \end{pmatrix} = \begin{pmatrix} W_0(t) \\ W_1(t) \\ \vdots \\ W_n(t) \end{pmatrix}$$

Suppose that a DC pension member contributes to an index bond and n kinds of financial products such as stocks or funds with the fixed contribute rate c (> 0). Then the portfolio weight process $\pi(t)$ is assumed to be

$$\pi(t) = \begin{pmatrix} \pi_0(t) \\ \pi_1(t) \\ \vdots \\ \pi_n(t) \end{pmatrix}$$

with

$$\sum_{k=0}^{n} \pi_k(t) < 1, \quad \pi_k(t) \geq 0 \text{ for all } t \text{ and } k = 0, 1, 2, \ldots, n$$

The first element $\pi_0(t)$ corresponds to the index bond, and $(\pi_1(t), \pi_2(t), \ldots, \pi_n(t))$ the n kinds of financial products. The member invests share of $1 - \sum_{i=0}^{n} \pi_i$ in the risk-free bond. Hence we define the wealth process $X^\pi(t)$ with an initial value of x ($0 \leq x < \infty$) in the following manner.

$$dX^\pi(t) = X^\pi(t)\left[Rdt + \pi^\top(t)\sigma\left(\theta dt + dW(t)\right)\right] + cY(t)dt \tag{2}$$

where $cY(t)$ is the amount of money contributed to the index bond and the financial products at time t. Assuming that the contribution is made continuously and $Y(t) > 0$ for all t, we set the stochastic discount factor $H(t)$ by

$$H(t) = \exp\left\{-Rt - \frac{1}{2}\|\theta\|^2 t - \theta^\top W(t)\right\}$$

which adjusts for the nominal interest rate and the market price of risk. The following definition is quoted from [2]

Definition 1 *A portfolio vector $\pi(t)$ is said to be admissible if the corresponding wealth process $X^\pi(t)$ in (2) satisfies*

$$X^\pi(t) + \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right] \geq 0, \ \mathbb{P} - a.s.$$

where $\{\mathcal{F}(t)\}_t$ denotes the Brownian filtration and $\mathbb{E}_t$ denotes the conditional expectation given $\mathcal{F}(t)$ under $\mathbb{P}$. We call the term

$$\mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right]$$

the human capital.

Let $\mathcal{A}_Y(x)$ be the class of all admissible portfolio process with an initial value x of $X^\pi(t)$, and the utility function of the CRRA (Constant Relative Risk Aversion) be the form

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma} \tag{3}$$

Then, as is discussed in [3], the representative pension plan member wishes to maximize the expected utility from the terminal value of the pension fund given an initial investment of $x > 0$, that is

$$\max_{\pi \in \bar{\mathcal{A}}(x)} \mathbb{E}\left[u(X^\pi(T))\right] \tag{4}$$

subject to

$$dX^\pi(t) = X^\pi(t)\left[Rdt + \pi^\top(t)\sigma\left(\theta dt + dW(t)\right)\right] + cY(t)dt \tag{5}$$

$$X^\pi(0) = x \tag{6}$$

where

$$\bar{\mathcal{A}}(x) = \{\pi \in \mathcal{A}_Y(x) : \mathbb{E}\left[u^-(X^\pi(T))\right] < \infty\}$$
$$u^-(\cdot) = \max\{-u(\cdot), 0\}$$

The problem stated in (4)-(6) is the classical terminal wealth optimization problem which expects that there is an additional term $cY(t)dt$ in the stochastic differential equation for the wealth process. Our final aim is to derive the optimal portfolio weight process $\pi(t)$ as the solution to the optimization problem.

2.2 Modeling for optimal investment problem

We consider the plan member's future contribution. For that purpose, define the present value of expected future contribution process by

$$D(t) = \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right] \tag{7}$$

then (7) is rewritten as

$$D(t) = \frac{c}{\beta}\left(\exp\{\beta(T-t)\} - 1\right) Y(t) \tag{8}$$

for $t \in [0, T]$ with

$$\beta = \kappa - R - \sigma_s\theta_0$$

The derivation from (7) to (8) is given in Appendix A. The representation (8) yields the stochastic differential equation (see Appendix B for the derivation).

$$dD(t) = D(t)\left[(R + \sigma_s\theta_0)\,dt + \sigma_Y^\top dW(t)\right] - cY(t)dt \tag{9}$$

Add (5) and (9) to cancel out the term $cY(t)$, then we obtain the total wealth process

$$V(t) = X^\pi(t) + D(t) \tag{10}$$

It follows from (5), (9) and (10) that

$$\begin{aligned} dV(t) &= dX^\pi(t) + dD(t) \\ &= X^\pi(t)\left[Rdt + \pi^\top(t)\sigma\,(\theta dt + dW(t))\right] + cY(t)dt \\ &\quad + D(t)\left[(R + \sigma_s\theta_0)\,dt + \sigma_Y^\top dW(t)\right] - cY(t)dt \\ &= V(t)\left[Rdt + \frac{X^\pi(t)\pi^\top(t)\sigma + D(t)\sigma_Y^\top}{V(t)}\,(\theta dt + dW(t))\right] \end{aligned} \tag{11}$$

where the equality $\sigma_s\theta_0 = \sigma_Y^\top\theta$ has been used in (11). We note that

$$V(0) = X^\pi(0) + D(0) = x + d \tag{12}$$

$$V(T) = X^\pi(T) + D(T) = X^\pi(T) \tag{13}$$

hold.

We consider also that the differential discounted process of the total wealth. By Ito product rule

$$\begin{aligned} d(H(t)V(t)) &= H(t)dV(t) + V(t)dH(t) + dV(t)dH(t) \\ &= H(t)V(t)\left[Rdt + \frac{X^\pi(t)\pi^\top(t)\sigma + D(t)\sigma_Y^\top}{V(t)}\,(\theta dt + dW(t))\right] \\ &\quad - H(t)V(t)\left(Rdt + \theta^\top dW(t)\right) - H(t)V(t)\frac{X^\pi(t)\pi^\top(t)\sigma + D(t)\sigma_Y^\top}{V(t)}\theta dt \\ &= H(t)V(t)\left[\frac{X^\pi(t)\pi^\top(t)\sigma + D(t)\sigma_Y^\top}{V(t)} - \theta^\top\right] dW(t) \\ &= H(t)V(t)\left[\frac{X^\pi(t)\sigma^\top\pi(t) + D(t)\sigma_Y}{V(t)} - \theta\right]^\top dW(t) \end{aligned} \tag{14}$$

The stochastic differential equation (11) with (12) and (13) shows that the classical terminal wealth optimization problem of (4)-(6) is equivalent to maximize $\mathbb{E}\left[u(V(T))\right]$ over a class of admissible portfolio processes $\pi(t)$.

2.3 Theoretical optimal investment solution

In view of the observation in the previous subsection, we restate the classical terminal wealth optimization problem (4)-(6) in terms of $V(t)$.

$$\max_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[u(X^{\pi}(T))\right] = \max_{\pi\in\mathcal{A}_1(x+d)} \mathbb{E}\left[u(V(T))\right] \tag{15}$$

$$dV(t) = V(t)\left[Rdt + \frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)}\left(\theta dt + dW(t)\right)\right] \tag{16}$$

where

$$V(t) = X^{\pi}(t) + D(t) \geq 0 \quad \text{for all } t \in [0, T]$$

$$\mathcal{A}_1(x+d) = \{\pi \in \mathcal{A}(x+d) : \mathbb{E}\left[u^{-}(V(T))\right] < \infty\} \tag{17}$$

and $\mathcal{A}(x+d)$ is the class of admissible portfolio process with an initial value $x+d$. Thus the problem (4)-(6) is replaced by (16)-(17).

Let the superscribe $(^*)$ be the corresponding optimal process hereafter, then the optimal wealth process is given by

$$V^*(t) = \frac{1}{H(t)}\mathbb{E}_t\left[(H(t))B^*\right]$$

with

$$B^* = (u')^{-1}(\lambda H(T))$$

where λ is the Lagrangian multiplier to be determined by the constraint

$$\mathbb{E}\left[H(T)B^*\right] = x + d$$

For the choice of the CRRA utility in particular, we have

$$B^* = (x+d)\frac{(H(T))^{-\frac{1}{\gamma}}}{\mathbb{E}\left[H(T))^{\frac{\gamma-1}{\gamma}}\right]}$$

and the corresponding optimal total wealth process has the form

$$V^*(t) = \frac{x+d}{H(t)}\frac{\mathbb{E}_t\left[H(T))^{\frac{\gamma-1}{\gamma}}\right]}{\mathbb{E}\left[H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{18}$$

Put

$$Z_1(t) = \exp\left\{\frac{1-\gamma}{\gamma}\theta^{\top}W(t) - \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2 ||\theta||^2 t\right\} \tag{19}$$

and set a deterministic function

$$f_1(t) = \exp\left\{\frac{1-\gamma}{\gamma}\left(R + \frac{1}{2\gamma}||\theta||^2\right)t\right\} \tag{20}$$

Then we obtain

$$(H(t))^{\frac{\gamma-1}{\gamma}} = f_1(t)Z_1(t) \tag{21}$$

and (18) becomes

$$V^*(t) = \frac{x+d}{H(t)}Z_1(t) \tag{22}$$

The representation (22) yields that

$$d(H(t)V^*(t)) = H(t)V^*(t)\frac{1-\gamma}{\gamma}\theta^\top dW(t) \tag{23}$$

The proof of the equality (23) is described in Appendix C. Denote $(X^\pi)^* \equiv X^*$ for simplicity, then it follows from (14) that

$$d(H(t)V^*(t)) = H(t)V^*(t)\left[\frac{X^*(t)\sigma^\top\pi^*(t) + D(t)\sigma_Y}{V^*(t)} - \theta^\top\right]^\top dW(t) \tag{24}$$

Comparing (23) with (24), we have

$$\frac{1-\gamma}{\gamma}\theta^\top = \frac{X^*(t)\sigma^\top\pi^*(t) + D(t)\sigma_Y}{V^*(t)} - \theta^\top$$

Therefore solving for $\pi^*(t)$ gives

$$\pi^*(t) = \frac{1}{\gamma}\left(\sigma^\top\right)^{-1}\theta\frac{V^*(t)}{X^*(t)} - \left(\sigma^\top\right)^{-1}\sigma_Y\frac{D(t)}{X^*(t)}$$

Since $X^*(t) = V^*(t) - D(t)$,

$$\pi^*(t) = \frac{1}{\gamma}\left(\sigma^\top\right)^{-1}\theta + \left(\sigma^\top\right)^{-1}\left(\frac{1}{\gamma}\theta - \sigma_Y\right)\frac{D(t)}{V^*(t) - D(t)}$$

which is the optimal weight process of our aim.

3 Numerical work

A result of basic numerical works will be presented on the day.

Appendices

Appendix A

Since

$$D(t) = \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)}cY(s)ds \mid \mathcal{F}(t)\right] = cY(t)\,\mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)}\frac{Y(s)}{Y(t)}ds \mid \mathcal{F}(t)\right]$$

we compute $\frac{H(s)}{H(t)}$ and $\frac{Y(s)}{T(t)}$. Each term is rewritten as

$$\begin{aligned}\frac{H(s)}{H(t)} &= \exp\left\{-R(s-t) - \frac{1}{2}\|\theta\|^2(s-t) - \theta^\top(W(s) - W(t))\right\}\\ \frac{Y(s)}{T(t)} &= \exp\left\{\kappa(s-t) - \frac{1}{2}\|\sigma_Y\|^2(s-t) + \sigma_Y^\top(W(s) - W(t))\right\}\end{aligned}$$

respectively. By applying the property $\frac{H(s)}{H(t)}, \frac{Y(s)}{T(t)} \perp\!\!\!\perp \mathcal{F}(t)$ for $s \geq t$,

$$D(t) = cY(t)g(t,T) \tag{25}$$

where

$$g(t,T) = \mathbb{E}\left[\int_0^{T-t} H(s)\frac{Y(s)}{Y(0)}ds\right] \tag{26}$$

Computing the integrand of (26), we have

$$
\begin{aligned}
H(s)\frac{Y(s)}{Y(0)} &= \exp\{(\kappa - R)s\}\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\theta\|^2 + \|\sigma_Y\|^2\right)s\right\} \\
&= \exp\{\beta s\}\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\}
\end{aligned}
$$

so hence

$$
\mathbb{E}\left[H(s)\frac{Y(s)}{Y(0)}\right] = \mathbb{E}\left[\exp\{\beta s\}\right]\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\}\right] \tag{27}
$$

When $0 \le t \le s$,

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\} \mid \mathcal{F}(t)\right] \\
&\quad = \exp\left\{(\sigma_Y-\theta)^\top W(t) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\}\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top (W(s)-W(t))\right\} \mid \mathcal{F}(t)\right]
\end{aligned}
$$

We can write

$$
\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top (W(s)-W(t))\right\} \mid \mathcal{F}(t)\right] = \mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top (W(s)-W(t))\right\}\right]
$$

Recall that $W(t) - W(s) \sim N(0, t-s)$, then the moment generating function of normal distribution implies

$$
\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top (W(s)-W(t))\right\}\right] = \exp\left\{\frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)(t-s)\right\}
$$

Therefore

$$
\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\} \mid \mathcal{F}(t)\right] = \exp\left\{(\sigma_Y-\theta)^\top W(t) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)t\right\}
$$

which shows that the process

$$
\left\{\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\}\right\}
$$

forms an exponential martingale. Then

$$
\mathbb{E}\left[\exp\{\beta s\}\right]\mathbb{E}\left[\exp\left\{(\sigma_Y-\theta)^\top W(s) - \frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s\right\}\right] = \exp\{\beta s\} \tag{28}
$$

By combining (25), (26), (27), and (28), together with the application of Fubini's theorem,

$$
D(t) = cY(t)\int_0^{T-t}\mathbb{E}\left[H(s)\frac{Y(s)}{Y(0)}\right]ds = \frac{c}{\beta}\left(\exp\{\beta(T-t)\} - 1\right)Y(t)
$$

which is (8). □

Appendix B

Noting that

$$
\begin{aligned}
dD(t) &= d\left(\frac{c}{\beta}\left(\exp\{\beta(T-t)\} - 1\right)Y(t)\right) \\
&= \frac{c}{\beta}\left(\exp\{\beta(T-t)\} - 1\right)dY(t) + \frac{c}{\beta}Y(t)d\left(\exp\{\beta(T-t)\} - 1\right)
\end{aligned} \tag{29}
$$

we have only to compute $dY(t)$. Put

$$I(t) = \left(\kappa - \frac{1}{2}||\sigma_Y||^2\right) t + \sigma_Y^\top W(t)$$

then, according to Ito-Doeblin formula,

$$dI(t) = \left(\kappa - \frac{1}{2}||\sigma_Y||^2\right) dt + \sigma_Y^\top dW(t)$$
$$dI(t)dI(t) = ||\sigma_Y^\top||dt$$

By setting the function $f(x) = \exp\{x\}$ moreover, the application of Ito-Doeblin formula yields

$$\begin{aligned} dY(t) &= df(I(t)) \\ &= f'(I(t))dI(t) + \frac{1}{2}f''(x)(I(t))dI(t)dI(t) \\ &= Y(t)\left(\kappa dt + \sigma_Y^\top dW(t)\right) \end{aligned} \tag{30}$$

Combining (29) and (30), we obtain

$$dD(t) = \frac{1}{\beta}\left(\exp\{\beta(T-t)\} - 1\right) cY(t)\left\{(\kappa - \beta)dt + \sigma_Y^\top dW(t)\right\} - cY(t)dt$$

As a consequence, using $\beta = \kappa - R - \sigma_s\theta_0$ ensures

$$dD(t) = D(t)\left[(R + \sigma_s\theta_0)\, dt + \sigma_Y^\top dW(t)\right] - cY(t)dt \qquad \square$$

Appendix C

According to Ito product rule,

$$d(H(t)V^*(t)) = H(t)dV^*(t) + V^*(t)dH(t) + dH(t)dV^*(t) \tag{31}$$

Let us write

$$H_1(t) = \frac{1}{H(t)} = \exp\left\{-\left(-Rt - \frac{1}{2}||\theta||^2 t - \theta^\top W(t)\right)\right\} \tag{32}$$

then $V^*(t)$ is rewritten as

$$V^*(t) = (x+d)H_1(t)Z_1(t)$$

Hence Ito product rule yields

$$\begin{aligned} dV^*(t) &= (x+d)d(H_1(t)Z_1(t)) \\ &= (x+d)\{H_1(t)dZ_1(t) + Z_1(t)dH_1(t) + dH_1(t)dZ_1(t)\} \end{aligned} \tag{33}$$

The first thing, we consider $dH_1(t)$. Put

$$J(t) = -Rt - \frac{1}{2}||\theta||^2 t - \theta^\top W(t)$$

then

$$H_1(t) = g(J(t))$$

with $g(x) = e^{-x}$ and

$$dJ(t) = -Rdt - \frac{1}{2}||\theta||^2 dt - \theta^\top dW(t)$$
$$dJ(t)dJ(t) = ||\theta||^2 dt$$

Since $g'(x) = -e^{-x}$ and $g''(x) = e^{-x}$,

$$\begin{aligned} dH_1(t) &= g'(J(t))dJ(t) + \frac{1}{2}g''(J(t))dJ(t)dJ(t) \\ &= H_1(t)\left(Rdt + ||\theta||^2 dt + \theta^\top dW(t)\right) \end{aligned} \tag{34}$$

by Ito-Doeblin formula.

We calculate $dZ_1(t)$ next. Since $Z_1(t)$ has the form

$$Z_1(t) = \exp\left\{\frac{1-\gamma}{\gamma}\theta^\top W(t) - \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2 ||\theta||^2 t\right\} \tag{35}$$

When we put

$$h(x) = e^x$$
$$K(t) = \left\{\frac{1-\gamma}{\gamma}\theta^\top W(t) - \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2 ||\theta||^2 t\right\}$$

similar argument to the case $H_1(t)$ derives

$$dZ_1(t) = Z_1(t)\frac{1-\gamma}{\gamma}\theta^\top dW(t) \tag{36}$$

Thus it follows from (32), (33), (34), (35), and (36) that

$$\begin{aligned} dV^*(t) &= (x+d)\left\{H_1(t)Z_1(t)\frac{1-\gamma}{\gamma}\theta^\top dW(t) + H_1(t)Z_1(t)\left(Rdt + ||\theta||^2 dt + \theta^\top dW(t)\right) + H_1(t)Z_1(t)\frac{1-\gamma}{\gamma}||\theta||^2 dt\right\} \\ &= V^*(t)\left(Rdt + \frac{1}{\gamma}\theta^\top dW(t) + \frac{1}{\gamma}||\theta||^2 dt\right) \end{aligned}$$

Therefore

$$\begin{aligned} d(H(t)V^*(t)) &= H(t)V^*(t)\left(Rdt + \frac{1}{\gamma}\theta^\top dW(t) + \frac{1}{\gamma}||\theta||^2 dt\right) - H(t)V^*(t)\left(Rdt + \theta^\top dW(t)\right) \\ &\quad - H(t)V^*(t)\left(Rdt + \theta^\top dW(t)\right)\left(Rdt + \frac{1}{\gamma}\theta^\top dW(t) + \frac{1}{\gamma}||\theta||^2 dt\right) \\ &= H(t)V^*(t)\frac{1-\gamma}{\gamma}\theta^\top dW(t) \end{aligned}$$

by (31). □

References

[1] Steven E. Shreve. (2004) Stochastic calculus for finance ii continuous-time models, Springer

[2] Aihua Zhang. Cristian-Oliver Ewald. (2010) Optimal investment for a pension fund under inflation risk, Math Meth Oper Res 71:353-369

[3] Nicole El Karoui, Monique Jeanblanc-Picque. (1998) Optimization of consumption with labor income, Finance Stochastic 2, 409-440